# 1 Introduction and commonly studied ensembles

Properties of random variables and their sums, products, and nonlinear functions underlie much of modern science. Sometimes forms of universality arise in these properties. For instance, the limiting distribution of large sums of independent identically distributed random variables approaches only a few possible fixed point distributions depending on the tail properties of the distribution of the random variables in question, the most famous of which is the normal distribution arising from sums of random variables with finite second moment.

In many fields, random matrices play a similarly important role to random variables. Random matrices are collections of random variables in $N \times M$ arrays. Depending on the application, one may be interested in properties of their singular values, eigenvalues, or eigenvectors, and very often the object of research is the probability distribution of eigenvalues of a square random matrix. Remarkably, for many problems exact properties of this distribution can be calculated in a limit of $N, M \to \infty$, and sometimes even at finite size. And, like the central limit theorem, random matrices which share similar properties also share limiting distributions of eigenvalues. So, understanding these distributions in simple settings can inform their properties in more complex ones.

## 1.1 Objects of study

Joint probability distribution of eigenvalues

$$\rho(x_1, \ldots, x_N) \tag{1}$$

gives the probability density that all $N$ eigenvalues of a random matrix take specific values.

The principle object of study for large random matrices is the limiting spectral density $\rho$, which gives the probability distribution over individual eigenvalues in the limit of large matrices. This is defined by

$$\rho(x) = \lim_{N\to\infty} \overline{\frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i)}$$

where the $x_i$ are the eigenvalues of a particular $N \times N$ matrix drawn from the ensemble of interest. Remarkably, in a variety of settings this quantity can be computed analytically. It is also often interesting to understand two-point functions, which described the correlation between eigenvalues, like

$$\rho(x, x') = \lim_{N\to\infty} \overline{\frac{1}{N}\sum_{i,j=1}^{N} \delta(x - x_i)\delta(x' - x_j)}$$

The behavior of the two-point functions govern gap statistics, the distribution of differences between successive eigenvalues, which are also universal.

Often the spectral density is bounded by one or more edges where the support vanishes. In the vicinity of the edge at finite $N$ there are deviations from the infinite-$N$

behavior that often have a large-deviation form. For instance, the probability of seeing the maximum eigenvalue have the value $x_{\max}$ in the vicinity of an upper edge $x_+$ takes the form

$$P(x_{\max}) \propto e^{-N\Phi(x_{\max}-x_+)}$$

where $\Phi$ is a large deviation function. The behavior of this large deviation function is also universal.

One can also ask about properties of the eigenvectors $\boldsymbol{v}$. One important property is the inverse participation ratio

$$\mathcal{I} = \frac{\sum_{i=1}^N v_i^4}{\left(\sum_{i=1}^N v_i^2\right)^2} = \frac{\sum_{i=1}^N v_i^4}{\|\boldsymbol{v}\|^4} = \sum_{i=1}^N \hat{v}_i^4 \tag{2}$$

which carries information about localization of the given eigenvector. If a matrix is not Hermitian, the left and right eigenvectors in general are not the same, and the overlap between eigenvectors with the same eigenvalue can be of interest. Finally, if you have two correlated random matrices, one can ask about the correlations between the directions of their eigenvectors.

## 1.2 Symmetric or Hermitian random matrices

Symmetric or Hermitian random matrices arise in the study of random energy functions, either in classical or quantum settings. They are characterised by the property that $A_{ij} = A_{ji}^*$, with $A_{ij}$ real for symmetric and complex for Hermitian matrices. In classical systems, the typical context in which one encounters a matrix is due to the Hessian of an energy $H$, or

$$\mathrm{Hess}_{ij}\, H(\boldsymbol{x}) = \frac{\partial^2 H(\boldsymbol{x})}{\partial x_i \partial x_j} \tag{3}$$

which is a symmetric matrix due to the commutation of derivatives. If the energy $H$ is derived from a random function, or $\boldsymbol{x}$ is drawn randomly, then the elements of the Hessian will themselves be random variables. Properties of the Hessian are directly related to stability and excitations of a system about minima.

In quantum mechanics, the Hamiltonian is itself a linear operator with matrix representations, and if the Hamiltonian depends on randomness its matrix elements will likewise be random variables. The study of random matrices is often considered to have been started by Eugene Wigner who, trying to describe properties of the spectra of heavy nuclei, modelled their Hamiltonian by Hermitian matrices of independent Gaussian variables. In fact, because of universality, certain properties of the eigenvalue distribution, like the distribution of gaps between subsequent eigenvalues, were accurately reproduced!

There are 10 universality classes of Hermitian random matrices, most of which are of interest specifically in quantum systems [1]. The limiting spectral density for several of these classes is the Wigner semicircle distribution. In the simplest case, where the matrix has Gaussian independent entries, the matrix ensemble is known as the Gaussian Orthogonal Ensemble (GOE).

## 1.3 Covariance matrices

Rectangular random matrices are common in the form of data. For instance, if one samples an $N$-dimensional vector $\boldsymbol{x}$ at a series of $M$ times, then one has a $N \times M$ matrix

$$X = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_t \end{bmatrix} \tag{4}$$

describing the full dataset. The vector $\boldsymbol{x}$ could described the position of a particle, or the values of a collection of stocks. Often one is interested in the covariance of $\boldsymbol{x}$, defined by

$$C_{ij} = \overline{x_i x_j} \tag{5}$$

if $\boldsymbol{x}$ has zero mean. The covariance can be estimated from the time series by

$$\tilde{C} = XX^T \tag{6}$$

This is also a symmetric random matrix, but it has properties different from generic symmetric matrices discussed in the previous section.

The limiting spectral density of $\tilde{C}$ if the vectors $\boldsymbol{x}$ are Gaussian with covariance $\Sigma$ is given by the Wishart distribution.

## 1.4 Asymmetric or non-Hermitian random matrices

Asymmetric random matrices arise in diverse settings, including phenomenological descriptions of ecosystems, couplings of neural networks, open quantum systems, and generally when the evolution of a system is not linked to the gradient of a potential energy. Such matrices generally have complex eigenvalues and cannot be diagonalized. The Lotka–Volterra equation for the populations of two interacting species is

$$\frac{\partial n_1}{\partial t} = \mu_1 n_1 (1 - n_1) + A_{12} n_2 \tag{7}$$

$$\frac{\partial n_2}{\partial t} = \mu_2 n_2 (1 - n_2) + A_{21} n_1 \tag{8}$$

In the absence of interactions $A$ the evolution of both species is simple logistic growth. When the species interact, the situation changes depending on the values of the interactions. They could be mutually beneficial, with $A_{12}, A_{21} > 0$, or mutually harmful with $A_{12}, A_{21} < 0$, or they could be asymmetric (like predator and prey) with $A_{12} > 0 > A_{21}$. If instead we have many interacting species,

$$\frac{\partial n_i}{\partial t} = \mu_i n_i (1 - n_i) + \sum_j A_{ij} n_j \tag{9}$$

where now $A$ is a matrix encoding the interspecies interactions, with no good reason to have any particular symmetry.

There are 34 classes of non-Hermitian random matrix, mostly relevant again to quantum researchers [2]. The simplest class are real matrices with independent random entries, whose limiting eigenvalue spectrum follows the disk law. This ensemble of matrices in known is the Ginibre ensemble.

## 1.5 Sums and products of matrices

In regular statistics, sums and products of random variables produce new random variables whose governing distributions are given from those of the summands or factors by standard formulas. Random matrix theory provides similar formulas to derive the spectral density from the spectral densities of the component matrices. The theory of *free probability* generalizes the notion of independence so important for random variables to *freeness*, which can be thought of as a kind of independence for noncommutative variables (like matrices). If two matrices are free, then the spectral density of their sum or product can be calculated knowing only their spectral densities.

## 1.6 Dense versus sparse

The methods covered in this course will almost exclusively be applicable to dense random matrices, which have all or nearly all of their elements nonzero. However, sparse random matrices – with few nonzero elements – are often of interest. These can have two kinds of disorder: the disorder of the values of the nonzero entries, and the disorder of which entries are nonzero.

Motivating examples are interactions of actual ecosystems, where many organisms affect each other in negligible ways, and in finite-dimensional systems, where interactions are mediated by short range and therefore distant elements have negligible interaction.

Sparse matrices are much more difficult to treat than dense ones, and see qualitatively different behavior. In certain circumstances they can be treated exactly, usually when the graph formed by joining interacting elements with edges has no loops or the size of those loops diverges with the system size. Then techniques from spin-glass theory known as the cavity method and approximate message passing can be used to understand their properties.

# References

[1] A. Altland and M. R. Zirnbauer, "Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures", Physical Review B **55**, 1142–1161 (1997).

[2] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, "Symmetry and topology in non-Hermitian physics", Physical Review X **9**, 041015 (2019).