2 Introducing Gaussian Orthogonal Ensemble and its joint probability distribution of eigenvalues

The Gaussian Orthogonal Ensemble (GOE) is the simplest ensemble of symmetric random matrices. The simplest *asymmetric* random matrix H is built from independently Gaussian distributed random variables $H_{ij} \sim \mathcal{N}(0,1)$ for all i and j. The notation $x \sim \mathcal{N}(\mu,\sigma^2)$ means that x is a random variable distributed like the normal distribution with mean μ and variance σ^2, or that its probability density function is

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} \tag{1}$$

but we want to start with the (simpler) analysis of symmetric matrices (whose eigenvalues are real) before moving on to asymmetric ones (whose eigenvalues are complex). To simplest way to build a symmetric matrix H_s from an asymmetric one is to sum the matrix and its transpose, or $H_s = \frac{1}{2}(H + H^T)$. How are the elements of such a symmetric matrix distributed? The diagonal is given by $(H_s)_{ii} = \frac{1}{2}(H_{ii}+H_{ii}) = H_{ii}$, so it is also unit normal with $(H_s)_{ii} \sim \mathcal{N}(0,1)$. The off-diagonal is given by one-half the sum of two independent Gaussian variables. Since the sum of independent Gaussians is another independent Gaussian with a variance given by the sum of the variances of the two variables, we have

$$\overline{(H_s)^2_{ij}} = \frac{1}{4}\overline{(H_{ij} + H_{ji})^2} = \frac{1}{4}2 = \frac{1}{2} \tag{2}$$

or $(H_s)_{ij} \sim \mathcal{N}(0,\frac{1}{2})$. What is the joint probability distribution over all the elements? Because they are independent, it is simply the product over the individual distributions, with

$$\rho(H_s) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(H_s)^2_{ii}} \prod_{i<j} \frac{1}{\sqrt{\pi}} e^{-(H_s)^2_{ij}} \tag{3}$$

where we have taken care to write dependence on only the off-diagonal elements in the upper triangle of H_s, since the upper and lower triangles are not independent.

This ensemble is emphatically not made of orthogonal matrices, so why does it have this name? The answer is that it has an invariance under the action of orthogonal matrices. Notice that in the PDF for H_s, the argument of the exponential can be written

$$\begin{aligned}
-\frac{1}{2}\sum_{i=1}^{N}(H_s)^2_{ii} - \sum_{i<j}(H_s)^2_{ij} &= -\frac{1}{2}\sum_{i=1}^{N}(H_s)^2_{ii} - \frac{1}{2}\sum_{i\neq j}(H_s)^2_{ij} = -\frac{1}{2}\sum_{i,j=1}^{N}(H_s)^2_{ij} \\
&= -\frac{1}{2}\sum_{i=1}^{N}(H_s H_s^T)_{ii} = -\frac{1}{2}\operatorname{Tr}(H_s H_s^T) = -\frac{1}{2}\operatorname{Tr} H_s^2
\end{aligned} \tag{4}$$

This means that

$$\rho(H_s) = \frac{1}{(2\pi)^{\frac{1}{4}N(N+1)}} e^{-\frac{1}{2}\operatorname{Tr} H_s^2} \tag{5}$$

Consider a transformed matrix OH_sO^{-1} for orthogonal O. Since

$$\mathrm{Tr}(OH_sO^{-1})^2 = \mathrm{Tr}\, OH_sO^{-1}OH_sO^{-1} = \mathrm{Tr}\, OH_sH_sO^{-1} = \mathrm{Tr}\, H_sH_sO^{-1}O = \mathrm{Tr}(H_s)^2 \tag{6}$$

it follows that $\rho(OH_sO^{-1}) = \rho(H_s)$. Therefore, any matrix is equally probable to its rotated version.

This is an example of an *invariant* ensemble. Invariant ensembles can be constructed by making the probability density function a function of only traces of powers of your matrix:

$$\rho(H) = f(\mathrm{Tr}\, H, \mathrm{Tr}\, H^2, \mathrm{Tr}\, H^3, \ldots) \tag{7}$$

which by construction has the same invariance. They have a special property that makes the study of their spectral properties much easier: because of their rotational invariance, the distribution of eigenvectors factorizes from the distribution of eigenvalues; the two are independent. We can see immediately why this is true: recall that

$$\mathrm{Tr}\, H = \sum_{i=1}^{N} x_i \tag{8}$$

the sum of the eigenvalues of H. Likewise,

$$\mathrm{Tr}\, H^n = \sum_{i=1}^{N} x_i^n \tag{9}$$

so any function only of traces is explicitly only a function of the eigenvalues, not the eigenvectors. In these ensembles, *any set of orthonormal eigenvectors is equally probable*.

We want to write down the joint probability distribution of eigenvalues for this ensemble. In more abstract terms, we want a charge of variables from the elements of H_s to the eigenvalues $\boldsymbol{x} = \{x_1, \ldots, x_N\}$ and eigenvectors $O = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N]$. A direct relationship between the two comes from the fact that symmetric matrices can be diagonalized using $H_s = OXO^T$ for X a matrix with diagonal given by $\boldsymbol{x}$. Then we would have

$$\rho(H_s)\, dH_s = \rho(H_s(\boldsymbol{x}, O))\, |\det J(H_s \to \{\boldsymbol{x}, O\}|\, d\boldsymbol{x}\, dO \tag{10}$$

where $J(H_s \to \{\boldsymbol{x}, O\})$ is the Jacobian of the transformation from H_s to $\boldsymbol{x}$ and O. We already know that, in invariant ensembles, the PDF $\rho(\boldsymbol{x}, O)$ does not depend on O (which is not true in general). But what about the Jacobian?

Formally differentiate H_s:

$$\delta H = (\delta O)XO^T + O(\delta X)O^T + OX(\delta O^T) \tag{11}$$

Now we need a little identity. Formally differentiating the identity gives

$$0 = \delta I = \delta(OO^T) = (\delta O)O^T + O(\delta O^T) \tag{12}$$

which implies $\delta O^T = -O^{-1}(\delta O)O^T = -O^T(\delta O)O^T$. Using this, we have

$$\delta H = (\delta O)XO^T + O(\delta X)O^T - OXO^T(\delta O)O^T \tag{13}$$

We simplify this by writing $\delta\hat{H} = O^T \delta H O$, or

$$\delta\hat{H} = O^T(\delta O)XO^T O + O^T O(\delta X)O^T O - O^T OXO^T(\delta O)O^T O = O^T(\delta O)X + \delta X - XO^T(\delta O) \tag{14}$$

Treating the Jacobian of $\delta\hat{H}$ is equivalent to that of δH since the two are related by an orthogonal transformation, and $|\det O| = 1$. Further define $\delta\Omega = O^T\delta O$, giving

$$\delta\hat{H} = (\delta\Omega)X - X(\delta\Omega) + \delta X \tag{15}$$

We can see that $\delta\Omega$ is antisymmetric, since

$$(\delta\Omega)^T = (O^T\delta O)^T = \delta O^T O = -O^T(\delta O)O^T O = -O^T\delta O = -\delta\Omega \tag{16}$$

Since X is diagonal, this is

$$\delta\hat{H}_{ij} = \sum_{k=1}^{N}(\delta\Omega_{ik}X_{kj} - X_{ik}\delta\Omega_{kj}) + \delta X_{ij} = (\delta\Omega_{ij}x_j - x_i\delta\Omega_{ij}) + \delta X_{ij} = \delta\Omega_{ij}(x_j - x_i) + dx_i\delta_{ij} \tag{17}$$

which gives

$$\frac{d\hat{H}_{ij}}{dx_k} = \delta_{ij}\delta_{ik} \qquad\qquad \frac{d\hat{H}_{ij}}{d\Omega_{k\ell}} = \delta_{ik}\delta_{j\ell}(x_j - x_i) \tag{18}$$

This is a bit weird to analyze, since we need the Jacobian of a matrix differentiated by a vector and a matrix. However, it's plausible to see that this transformation is diagonal in $\boldsymbol{x}$ and Ω, with N eigenvalues $x_1, \ldots, x_N$ and $\frac{1}{2}N(N-1)$ eigenvalues $x_j - x_i$, $i < j$. To increase plausibility, see chapter 7 of the text. Therefore the determinant of the transformation, being the product of the eigenvalues, is

$$|\det J(H_s \to \{\boldsymbol{x}, O\})| = \prod_{i<j}^{N} |x_j - x_i| = |\Delta_N(\boldsymbol{x})| \tag{19}$$

This object is known as the Vandermonde determinant, so-called because it can also be written

$$\Delta_N(\boldsymbol{x}) = \det_{ij}(x_j^{i-1}) = \det \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \\ x_1^2 & \cdots & x_N^2 \\ \vdots & \cdots & \vdots \\ x_1^{N-1} & \cdots & x_N^{N-1} \end{bmatrix} \tag{20}$$

So we have that the joint probability distribution for eigenvalues in the GOE is

$$\rho(\boldsymbol{x}) = \frac{1}{Z_N} e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2} \prod_{j<k} |x_j - x_k| \tag{21}$$

for

$$Z_N = \frac{(2\pi)^{\frac{1}{4}N(N+1)}}{\int dO} \tag{22}$$

with

$$\int dO = \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)} \qquad \Gamma_N(a) = \pi^{N(N-1)/4} \prod_{i=1}^{N} \Gamma(a-(i-1)/2) \tag{23}$$

If you're curious about the volume of the orthogonal group (which will not be relevant for our derivation of the asymptotic form of the spectral density) you can see chapter 6 of the text.

This material is in chapters 6 and 7 in the Livan text.