5 The resolvent several other ways, continued

Recall the resolvent

$$G(z) = \frac{1}{N}\overline{\mathrm{Tr}(zI - H)^{-1}} \tag{1}$$

and its relation to the spectral density:

$$\rho(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \mathrm{Im}\, G(x - i\epsilon) \tag{2}$$

The resolvent is a powerful tool to find the spectral density, partially because it can be calculated so many different ways. We saw last week a field theory approach to its calculation. Today we will see some other ways to compute it.

Supersymmetry

The supersymmetric method starts again from the identity

$$A^{-1}_{ij} = \sqrt{\frac{\det A}{(2\pi)^N}} \int d\boldsymbol{s}\, e^{-\frac{1}{2}\boldsymbol{s}^T A \boldsymbol{s}} s_i s_j \tag{3}$$

Again, we need to eliminate the extraneous determinant in front. Supersymmetry does this by introducing extra fields that cancel it. Consider the following counting problem: each time you do a Gaussian integral

$$\int d\boldsymbol{s}\, e^{-\frac{1}{2}\boldsymbol{s}^T A \boldsymbol{s}} = \sqrt{\frac{(2\pi)^N}{\det A}} \tag{4}$$

you get a factor of $\det A$ to the $-\frac{1}{2}$. On the other hand each time you do a Gaussian integral of Grassmann variables

$$\int d\bar{\boldsymbol{\eta}}\, d\boldsymbol{\eta}\, e^{-\bar{\boldsymbol{\eta}}^T A \boldsymbol{\eta}} = \det A \tag{5}$$

you get a factor of $\det A$ to the $+1$. Therefore, to have the correct number of $\det A$ we write two real-valued Gaussian integrals and one Grassmann Gaussian integral, or

$$\begin{aligned} A^{-1}_{ij} &= \int d\boldsymbol{s}\, \frac{d\hat{\boldsymbol{s}}}{(2\pi)^N}\, d\bar{\boldsymbol{\eta}}\, d\boldsymbol{\eta}\, e^{-\frac{1}{2}\boldsymbol{s}^T A \boldsymbol{s} - \frac{1}{2}\hat{\boldsymbol{s}}^T A \hat{\boldsymbol{s}} - \bar{\boldsymbol{\eta}}^T A \boldsymbol{\eta}} s_i s_j \\ &= \int d\boldsymbol{s}\, \frac{d\hat{\boldsymbol{s}}}{(2\pi)^N}\, d\bar{\boldsymbol{\eta}}\, d\boldsymbol{\eta}\, e^{-\frac{1}{2}\boldsymbol{s}^T A \boldsymbol{s} - \frac{1}{2}\hat{\boldsymbol{s}}^T A \hat{\boldsymbol{s}} - \bar{\boldsymbol{\eta}}^T A \boldsymbol{\eta}} \frac{1}{2}(s_i s_j + \hat{s}_i \hat{s}_j) \end{aligned} \tag{6}$$

where we have used the symmetry of the integral exchanging $\boldsymbol{s}$ and $\hat{\boldsymbol{s}}$. Why is this called supersymmetric? If we defined a superfield indexed by Grassmann indices $\bar{\theta}$ and θ by

$$\boldsymbol{\phi}(\theta) = \frac{1}{\sqrt{2}}(\boldsymbol{s} + i\hat{\boldsymbol{s}}) + \bar{\theta}\boldsymbol{\eta} + \bar{\boldsymbol{\eta}}\theta + \bar{\theta}\theta\frac{1}{\sqrt{2}}(\boldsymbol{s} - i\hat{\boldsymbol{s}}) \tag{7}$$

and define the measure

$$d\boldsymbol{\phi} = d\boldsymbol{s}\,\frac{d\hat{\boldsymbol{s}}}{(2\pi)^N}\,d\bar{\boldsymbol{\eta}}\,d\boldsymbol{\eta} \tag{8}$$

Then we can write A_{ij}^{-1} as

$$A_{ij}^{-1} = \int d\boldsymbol{\phi}\,e^{-\frac{1}{2}\int d\theta\,d\bar{\theta}\,\boldsymbol{\phi}(\theta)^T A\boldsymbol{\phi}(\theta)}\frac{1}{2}\int d\theta_1\,d\bar{\theta}_1\,d\theta_2\,d\bar{\theta}_2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\phi_i(1)\phi_j(2) \tag{9}$$

This integral has two pieces: the exponential piece, which can be written in an explicitly supersymmetric way (meaning that there is no explicit dependence on the indices θ and $\bar{\theta}$), and the prefactor

$$\frac{1}{2}\int d\theta_1\,d\bar{\theta}_1\,d\theta_2\,d\bar{\theta}_2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\phi_i(1)^T\phi_j(2) = \frac{1}{2}(s_i s_j + \hat{s}_i\hat{s}_j) \tag{10}$$

which is *not* supersymmetric. However, because the exponential *is* supersymmetric, our saddle point value of our order parameter will be supersymmetric.

Replacing A with $zI - H$ (and writing $d1 = d\theta_1\,d\bar{\theta}_1$, etc) gives

$$G(z) = \overline{\int d\boldsymbol{\phi}\,e^{-\frac{1}{2}\int d1\,\boldsymbol{\phi}(1)^T(zI-H)\boldsymbol{\phi}(1)}\frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\frac{\boldsymbol{\phi}(1)^T\boldsymbol{\phi}(2)}{N}} \tag{11}$$

Averaging over H gives

$$G(z) = \int d\boldsymbol{\phi}\,e^{-\frac{1}{2}z\int d1\,\boldsymbol{\phi}(1)^T\boldsymbol{\phi}(1)+\frac{1}{8N}\int d1\,d2\,[\boldsymbol{\phi}(1)^T\boldsymbol{\phi}(2)]^2}\frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\frac{\boldsymbol{\phi}(1)^T\boldsymbol{\phi}(2)}{N} \tag{12}$$

Now we introduce an order parameter $\mathbb{Q}(1,2) = \frac{1}{N}\boldsymbol{\phi}(1)^T\boldsymbol{\phi}(2)$. This gives

$$\begin{aligned}
G(z) &= \int d\boldsymbol{\phi}\,d\mathbb{Q}\,e^{-\frac{N}{2}z\int d1\,\mathbb{Q}(1,1)+\frac{N}{8}\int d1\,d2\,\mathbb{Q}(1,2)^2}\delta\!\left(\tfrac{1}{2}(N\mathbb{Q}(1,2)-\boldsymbol{\phi}(1)^T\boldsymbol{\phi}(2))\right)\\
&\quad\times\frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\mathbb{Q}(1,2)\\
&= \int d\mathbb{Q}\,e^{-\frac{N}{2}z\int d1\,\mathbb{Q}(1,1)+\frac{N}{8}\int d1\,d2\,\mathbb{Q}(1,2)^2}\int d\tilde{\mathbb{Q}}\,e^{\frac{N}{2}\int d1\,d2\,\tilde{\mathbb{Q}}(1,2)\mathbb{Q}(1,2)}\int d\boldsymbol{\phi}\,e^{-\frac{1}{2}\int d1\,d2\,\boldsymbol{\phi}(1)^T\tilde{\mathbb{Q}}(1,2)\phi(2)}\\
&\quad\times\frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\mathbb{Q}(1,2)\\
&= \int d\mathbb{Q}\,e^{-\frac{N}{2}z\int d1\,\mathbb{Q}(1,1)+\frac{N}{8}\int d1\,d2\,\mathbb{Q}(1,2)^2}\int d\tilde{\mathbb{Q}}\,e^{\frac{N}{2}\int d1\,d2\,\tilde{\mathbb{Q}}(1,2)\mathbb{Q}(1,2)-\frac{N}{2}\,\mathrm{sdet}\,\tilde{\mathbb{Q}}}\\
&\quad\times\frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1+\bar{\theta}_2\theta_2)\mathbb{Q}(1,2)
\end{aligned}$$

where we have exponentiated the δ function and integrated away the $\boldsymbol{\phi}$. Treating the integral in $\tilde{\mathbb{Q}}$ with a saddle point gives

$$0 = \frac{\partial\mathcal{S}}{\partial\tilde{\mathbb{Q}}(1,2)} = \frac{1}{2}\mathbb{Q}(1,2) + \frac{1}{2}\tilde{\mathbb{Q}}^{-1}(1,2) \tag{13}$$

which results in $\tilde{\mathbb{Q}}(1,2) = \mathbb{Q}^{-1}(1,2)$. Finally, the whole thing can be written

$$G(z) = \int d\mathbb{Q}\, e^{-\frac{N}{2}z\int d1\,\mathbb{Q}(1,1)+\frac{N}{8}\int d1\,d2\,\mathbb{Q}(1,2)^2+\frac{N}{2}\,\mathrm{sdet}\,\mathbb{Q}}\frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1 + \bar{\theta}_2\theta_2)\mathbb{Q}(1,2)$$

where we have used the fact that $\delta(1,1) = 0$ to eliminate the term coming from $\tilde{\mathbb{Q}}\mathbb{Q}$. Now we can write the saddle-point condition for $\mathbb{Q}$, with

$$0 = \frac{\partial \mathcal{S}}{\partial \mathbb{Q}(1,2)} = -\frac{1}{2}z\delta(1,2) + \frac{1}{4}Q(1,2) + \frac{1}{2}\mathbb{Q}^{-1}(1,2) \tag{14}$$

Convolving everything by $2\mathbb{Q}$ gives

$$0 = \frac{1}{2}\int d3\,\mathbb{Q}(1,3)\mathbb{Q}(3,2) - z\mathbb{Q}(1,2) + \delta(1,2) \tag{15}$$

A generic supersymmetric operator has the form $\mathbb{Q}(1,2) = q_d\delta(1,2) + q_0$. Inserting this and expanding everything, we find

$$\begin{aligned}
0 &= \frac{1}{2}\int d3\,(q_d\delta(1,3) + q_0)(q_d\delta(3,2) + q_0) - z(q_d\delta(1,2) + q_0) + \delta(1,2) \\
&= \frac{1}{2}(q_d^2\delta(1,2) + 2q_dq_0) - z(q_d\delta(1,2) + q_0) + \delta(1,2) \\
&= \left(\frac{1}{2}q_d^2 - zq_d + 1\right)\delta(1,2) + (q_dq_0 - zq_0)
\end{aligned}$$

Each term has to independently give zero. Notice that these give the exact same conditions on q_d and q_0 as the replica calculation when $n = 0$. Solving, we again find $q_0 = 0$, $q_d = z \pm \sqrt{z^2 - 2}$. Then, to evaluate G, we notice that for supersymmetric $\mathbb{Q}$,

$$\mathrm{sTr}\,\mathbb{Q} = \int d1\,\mathbb{Q}(1,1) = 0 \qquad \mathrm{sdet}\,\mathbb{Q} = 1 \tag{16}$$

$$\int d1\,d2\mathbb{Q}(1,2)^2 = \int d1\,d2\,(q_0^2 + q_dq_0\delta(1,2)) = \int d1\,q_dq_0\delta(1,1) = 0 \tag{17}$$

where in the last we used that $\delta(1,2)^2 = 0$ and $\delta(1,1) = 0$. Therefore, the entire argument of the exponential is identically zero at the saddle point value of $\mathbb{Q}$! This is in fact typical of supersymmetric integrals. As for the replicas, all our value comes from the prefactor, which gives

$$\begin{aligned}
G(z) &= \frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1 + \bar{\theta}_2\theta_2)\mathbb{Q}(1,2) \\
&= \frac{1}{2}\int d1\,d2\,(\bar{\theta}_1\theta_1 + \bar{\theta}_2\theta_2)(q_0 + q_d\delta(1,2)) \\
&= \frac{1}{2}\int d1\,(2\bar{\theta}_1\theta_1q_d + q_0) = q_d = z \pm \sqrt{z^2 - 2}
\end{aligned} \tag{18}$$

Once again we find the classic resolvent for the semicircle!

In the saddle-point approach, the supersymmetric technique is nearly identical to that of replicas. However, the great strength of supersymmetry is that it comes with a toolkit for working on systems beyond mean-field: we could have treated the integral over $\boldsymbol{\phi}$ as a regular supersymmetric field theory and make a perturbative expansion. Because of this, supersymmetry provides a toolkit for working at finite N or for sparse or otherwise structured systems.