

5 The resolvent several other ways, continued continued

Recall the resolvent

$$G(z) = \frac{1}{N} \overline{\text{Tr}(zI - H)^{-1}} \quad (1)$$

and its relation to the spectral density:

$$\rho(x) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \text{Im } G(x - i\epsilon) \quad (2)$$

The resolvent is a powerful tool to find the spectral density, partially because it can be calculated so many different ways. We saw last week a field theory approach to its calculation. Today we will see some other ways to compute it.

Dynamics

A final approach again starts from the observation that

$$A_{ij}^{-1} = \sqrt{\frac{\det A}{(2\pi)^N}} \int ds e^{-\frac{1}{2} s^T A s} s_i s_j = \frac{\int ds e^{-\frac{1}{2} s^T A s} s_i s_j}{\int ds e^{-\frac{1}{2} s^T A s}} = \langle s_i s_j \rangle \quad (3)$$

is simply the average of $s_i s_j$ over a Boltzmann distribution with energy $E(s) = \frac{1}{2} s^T A s$ and temperature $T = 1$ (the average over this distribution is written with angle brackets). We know that the Boltzmann distribution

$$\rho(s) = \frac{1}{Z} e^{-\beta E(s)} \quad (4)$$

is the stationary distribution of Langevin dynamics

$$\dot{s}(t) = -\nabla E(s(t)) + \xi(t) \quad (5)$$

where ξ is centered Gaussian noise with variance $\langle \xi_i(t) \xi_j(s) \rangle = 2T \delta_{ij} \delta(t - s)$ for $T = \beta^{-1}$. So, we can understand the trace of the inverse matrix $\text{Tr } A^{-1}$ by averaging the behavior of $\|s(t)\|^2$ under the Langevin dynamics above at large times. Our dynamics should have $T = 1$ and $A = zI - H$, so inserting E defined above gives

$$\dot{s}(t) = -zs(t) + Hs(t) + \xi(t) \quad (6)$$

This is the equation of N particles moving in a bath with an attraction to the origin and pairwise interactions mediated by the matrix H .

In fact, the resolvent is also encoded in other pieces of the theory. Consider the response function, defined by adding an external field $\mathbf{h}$ to the equation of motion and taking the average variation of the trajectory with respect to the external field, or

$$R(t, s) = \frac{1}{N} \left\langle \sum_{i=1}^N \frac{\delta s_i(t)}{\delta h_i(s)} \right\rangle \Big|_{\mathbf{h}=0} \quad (7)$$

We can write a self-consistent formula for the response by taking the variation of the equation of motion with respect to $\mathbf{h}$. Since ξ is independent of $\mathbf{h}$, this gives

$$\frac{\partial}{\partial t} \frac{\delta s_i(t)}{\delta h_j(s)} = - \sum_k (z\delta_{ik} - H_{ik}) \frac{\delta s_k(t)}{\delta h_j(s)} + \frac{\delta h_i(t)}{\delta h_j(s)} = - \sum_k (z\delta_{ik} - H_{ik}) \frac{\delta s_k(t)}{\delta h_j(s)} + \delta_{ij} \delta(t-s) \quad (8)$$

we briefly define a *matrix response function* $M_{ij}(t, s) = \langle \frac{\delta s_i(t)}{\delta h_j(s)} \rangle$, where $R(t, s) = \frac{1}{N} \text{Tr } M(t, s)$. Then, taking the average over noise of the previous equation, we have the matrix equation

$$\frac{\partial}{\partial t} M(t, s) = -(zI - H)M(t, s) + I\delta(t - s) \quad (9)$$

In the stationary state, the response function is time translation invariant, meaning that it is only a function of the difference between the perturbation and measurement, or $M(t + \tau, t) = M(\tau)$. Substituting this and making the Fourier transform of the result, we find

$$i\omega \hat{M}(\omega) = -(zI - H)\hat{M}(\omega) + I \quad (10)$$

which is solved to give

$$\hat{M}(\omega) = ((z + i\omega)I - H)^{-1} \quad (11)$$

which implies directly that

$$G(z) = \frac{1}{N} \text{Tr } \hat{M}(0) = \hat{R}(0) \quad (12)$$

So, the resolvent can be found from the zero-frequency value of the Fourier transform of the response function of our Langevin dynamics! We will see that this is an easier way to extract G than from the expectation of $\|s\|^2$.

To solve this, any existing technique for studying Langevin dynamics would suffice. Here, we use techniques from dynamical mean field theory (DMFT). Define the generating function

$$Z = \int_{s(0)=s_0} \mathcal{D}s \delta(\xi(t) - \dot{s}(t) - zs(t) + Hs(t)) \quad (13)$$

$$= \int \mathcal{D}s \mathcal{D}\hat{s} e^{\int dt \hat{s}(t)^T [\xi(t) - \dot{s}(t) - zs(t) + Hs(t)]} \quad (14)$$

where we have used a functional Lagrange multiplier.¹ Note that we can relate R to the

¹Our generating function has a functional δ function, but no Jacobian. We are correct in neglecting the functional Jacobian so long as we take the Itô convention for our path integral, which guarantees that the Jacobian is always one. This is because in that convention, when considering the matrix with rows of t and columns of t' , the matrix is only nonzero on the diagonal and on *one* of the neighboring diagonals, i.e., it is bidiagonal. Here the distinction between conventions is meaningless anyway.

average value of $\mathbf{s}(t) \cdot \hat{\mathbf{s}}(s)$. This is because

$$\begin{aligned}
R(t, s) &= \left\langle \frac{1}{N} \sum_{i=1}^N \frac{\delta s_i(t)}{\delta h_i(s)} \right\rangle \Big|_{\mathbf{h}=0} \\
&= \left\langle \frac{1}{N} \sum_{i=1}^N \frac{\delta}{\delta h_i(s)} \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} s_i(t) e^{\int dt \hat{\mathbf{s}}(t)^T [\xi(t) - \dot{\mathbf{s}}(t) - z\mathbf{s}(t) + H\mathbf{s}(t) + \mathbf{h}(t)]} \right\rangle \Big|_{\mathbf{h}=0} \\
&= \left\langle \frac{1}{N} \sum_{i=1}^N \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} s_i(t) \hat{s}_i(s) e^{\int dt \hat{\mathbf{s}}(t)^T [\xi(t) - \dot{\mathbf{s}}(t) - z\mathbf{s}(t) + H\mathbf{s}(t) + \mathbf{h}(t)]} \right\rangle \Big|_{\mathbf{h}=0} \\
&= \frac{1}{N} \langle \mathbf{s}(t) \cdot \hat{\mathbf{s}}(s) \rangle
\end{aligned} \tag{15}$$

First we average over the noise ξ , which gives

$$\begin{aligned}
\langle Z \rangle &= \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} e^{\int dt \hat{\mathbf{s}}(t)^T [-\dot{\mathbf{s}}(t) - z\mathbf{s}(t) + H\mathbf{s}(t)] + \frac{1}{2} \int dt ds \hat{\mathbf{s}}(t)^T \langle \xi(t) \xi(s)^T \rangle \hat{\mathbf{s}}(s)} \\
&= \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} e^{\int dt \hat{\mathbf{s}}(t)^T [-\dot{\mathbf{s}}(t) - z\mathbf{s}(t) + H\mathbf{s}(t)] + \int dt ds \hat{\mathbf{s}}(t)^T \delta(t-s) I \hat{\mathbf{s}}(s)} \\
&= \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} e^{\int dt \hat{\mathbf{s}}(t)^T [\hat{\mathbf{s}}(t) - \dot{\mathbf{s}}(t) - z\mathbf{s}(t) + H\mathbf{s}(t)]}
\end{aligned} \tag{16}$$

Now we average over the matrix H , which gives

$$\begin{aligned}
\overline{e^{\int dt \hat{\mathbf{s}}(t)^T H \mathbf{s}(t)}} &= e^{\frac{1}{2} \sum_{ijkl} \int dt \hat{s}_i(t) s_j(t) \int ds \hat{s}_k(s) s_l(s) \overline{H_{ij} H_{kl}}} \\
&= e^{\frac{1}{4N} \sum_{ijkl} \int dt \hat{s}_i(t) s_j(t) \int ds \hat{s}_k(s) s_l(s) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})} \\
&= e^{\frac{N}{4} \int dt ds \left(\frac{\mathbf{s}(t) \cdot \mathbf{s}(s)}{N} \frac{\hat{\mathbf{s}}(t) \cdot \hat{\mathbf{s}}(s)}{N} + \frac{\mathbf{s}(t) \cdot \hat{\mathbf{s}}(s)}{N} \frac{\hat{\mathbf{s}}(t) \cdot \mathbf{s}(s)}{N} \right)}
\end{aligned} \tag{17}$$

Now we see that once again, everything in our integrand is a function only of scalar products

$$C(t, s) = \frac{1}{N} \mathbf{s}(t) \cdot \mathbf{s}(s) \quad R(t, s) = \frac{1}{N} \mathbf{s}(t) \cdot \hat{\mathbf{s}}(s) \quad D(t, s) = \frac{1}{N} \hat{\mathbf{s}}(t) \cdot \hat{\mathbf{s}}(s) \tag{18}$$

which we must insert with more functional δ functions:

$$\begin{aligned}
\overline{\langle Z \rangle} &= \int \mathcal{D}\mathbf{C} \mathcal{D}\mathbf{R} \mathcal{D}\mathbf{D} e^{N \int dt ds [D(t, s) - (z + \partial_t) R(t, s)] \delta(t-s) + \frac{N}{4} \int dt ds (C(t, s) D(t, s) + R(t, s) R(s, t))} \\
&\quad \times \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} \delta(NC(t, s) - \mathbf{s}(t)^T \mathbf{s}(s)) \delta(NR(t, s) - \mathbf{s}(t)^T \hat{\mathbf{s}}(s)) \delta(ND(t, s) - \mathbf{s}(t)^T \mathbf{s}(s)) \\
&= \int \mathcal{D}\mathbf{C} \mathcal{D}\mathbf{R} \mathcal{D}\mathbf{D} e^{N \int dt ds [D(t, s) - (z + \partial_t) R(t, s)] \delta(t-s) + \frac{N}{4} \int dt ds (C(t, s) D(t, s) + R(t, s) R(s, t))} \\
&\quad \times \int \mathcal{D}\mathbf{s} \mathcal{D}\hat{\mathbf{s}} \mathcal{D}\tilde{\mathbf{C}} \mathcal{D}\tilde{\mathbf{R}} \mathcal{D}\tilde{\mathbf{D}} e^{\frac{1}{2} \int dt ds \text{Tr} \begin{bmatrix} \tilde{\mathbf{C}}(t, s) & \tilde{\mathbf{R}}(t, s) \\ \tilde{\mathbf{R}}(s, t) & \tilde{\mathbf{D}}(s, t) \end{bmatrix} \begin{bmatrix} \mathbf{C}(t, s) & \mathbf{R}(t, s) \\ \mathbf{R}(s, t) & \mathbf{D}(s, t) \end{bmatrix}} \\
&\quad \times e^{-\frac{1}{2} \int dt ds \begin{bmatrix} \mathbf{s}(t) \\ \hat{\mathbf{s}}(t) \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{C}}(t, s) & \tilde{\mathbf{R}}(t, s) \\ \tilde{\mathbf{R}}(s, t) & \tilde{\mathbf{D}}(s, t) \end{bmatrix} \begin{bmatrix} \mathbf{s}(s) \\ \hat{\mathbf{s}}(s) \end{bmatrix}}
\end{aligned}$$

Integrating away the s and $\hat{s}$ gives

$$\begin{aligned} \overline{\langle Z \rangle} &= \int \mathcal{D}C \mathcal{D}R \mathcal{D}D e^{N \int dt ds [D(t,s) - (z + \partial_t)R(t,s)] \delta(t-s) + \frac{N}{4} \int dt ds (C(t,s)D(t,s) + R(t,s)R(s,t))} \\ &\quad \times \int \mathcal{D}\tilde{C} \mathcal{D}\tilde{R} \mathcal{D}\tilde{D} e^{\frac{1}{2} \int dt ds \text{Tr} \begin{bmatrix} \tilde{C}(t,s) & \tilde{R}(t,s) \\ \tilde{R}(s,t) & \tilde{D}(t,s) \end{bmatrix} \begin{bmatrix} C(t,s) & R(t,s) \\ R(s,t) & D(t,s) \end{bmatrix} - \frac{1}{2} \log \det \begin{bmatrix} \tilde{C}(t,s) & \tilde{R}(t,s) \\ \tilde{R}(s,t) & \tilde{D}(t,s) \end{bmatrix}} \end{aligned}$$

Eliminating the tilde fields with a saddle point in the usual way gives

$$\begin{aligned} \overline{\langle Z \rangle} &\propto \int \mathcal{D}C \mathcal{D}R \mathcal{D}D e^{N \int dt ds [D(t,s) - (z + \partial_t)R(t,s)] \delta(t-s) + \frac{N}{4} \int dt ds (C(t,s)D(t,s) + R(t,s)R(s,t))} \\ &\quad \times e^{+\frac{1}{2} \log \det \begin{bmatrix} C(t,s) & R(t,s) \\ R(s,t) & D(t,s) \end{bmatrix}} \end{aligned}$$

where we don't consider the divergent functional trace of the identity. Note that this is the typical form of a DMFT with the self-energies $\Sigma(t,s) = R(t,s)$ and $D(t,s) = C(t,s)$. Now we are ready to solve for the order parameters using a saddle point! For reasons we will see, it is most convenient to consider the derivative with respect to the matrix

$$Q(t,s) = \begin{bmatrix} C(t,s) & R(t,s) \\ R(s,t) & D(t,s) \end{bmatrix} \quad (19)$$

Taking such a derivative, we should treat $R(t,s)$ and $R(s,t)$ as independent variables. To facilitate this, we quickly rewrite

$$\begin{aligned} &N \int dt ds [D(t,s) - (z + \partial_t)R(t,s)] \delta(t-s) \\ &= N \int dt ds [D(t,s) - \frac{1}{2}(z + \partial_t)R(t,s) - \frac{1}{2}(z + \partial_s)R(s,t)] \delta(t-s) \\ &= N \int dt ds D(t,s) \delta(t-s) - \frac{N}{2} \int dt ds \left[[(z - \partial_t)\delta(t-s)]R(t,s) + \frac{1}{2}[(z + \partial_t)\delta(t-s)]R(s,t) \right] \end{aligned}$$

where we have used

$$\int dx \delta'(x-y)f(x) = -f'(y) \quad (20)$$

Now we can take derivatives with respect to Q , which give

$$0 = \frac{\delta S}{\delta Q(t,s)} = \begin{bmatrix} \frac{1}{4}D(t,s) & \frac{1}{2}(z + \partial_t)\delta(t-s) + \frac{1}{4}R(s,t) \\ \frac{1}{2}(z + \partial_t)\delta(t-s) + \frac{1}{4}R(t,s) & \delta(t-s) + \frac{1}{4}C(t,s) \end{bmatrix} + \frac{1}{2}Q^{-1}(t,s) \quad (21)$$

Rather than mess with the functional inverse matrix of Q , we remove it by convolving it to both sides, which gives

$$0 = \int du \begin{bmatrix} \frac{1}{4}D(t,u) & \frac{1}{2}(z - \partial_t)\delta(t-u) + \frac{1}{4}R(u,t) \\ \frac{1}{2}(z + \partial_t)\delta(t-u) + \frac{1}{4}R(t,u) & \delta(t-u) + \frac{1}{4}C(t,u) \end{bmatrix} Q(u,s) + \frac{1}{2}\delta(t,s) \quad (22)$$

which gives the four equations left to right, top to bottom,

$$\begin{aligned}
0 &= \frac{1}{2}(z - \partial_t)R(s, t) + \frac{1}{4} \int du (D(t, u)C(u, s) + R(s, u)R(u, t)) + \frac{1}{2}\delta(t - s) \\
0 &= \frac{1}{2}(z - \partial_t)D(t, s) + \frac{1}{4} \int du (D(t, u)R(u, s) + R(u, t)D(u, s)) \\
0 &= \frac{1}{2}(z + \partial_t)C(t, s) + R(s, t) + \frac{1}{4} \int du (R(t, u)C(u, s) + C(t, u)R(s, u)) \\
0 &= \frac{1}{2}(z + \partial_t)R(t, s) + D(t, s) + \frac{1}{4} \int du (R(t, u)R(u, s) + D(t, u)D(u, s)) + \frac{1}{2}\delta(t - s)
\end{aligned}$$

First, notice that the equation for D is consistent with $D = 0$ for all times. Focusing then on the last two equations, we have

$$\begin{aligned}
0 &= \frac{1}{2}(z + \partial_t)C(t, s) + R(s, t) + \frac{1}{4} \int du (R(t, u)C(u, s) + C(t, u)R(s, u)) \\
0 &= \frac{1}{2}(z + \partial_t)R(t, s) + \frac{1}{4} \int du R(t, u)R(u, s) + \frac{1}{2}\delta(t - s)
\end{aligned}$$

We seek solutions to these equations in the stationary state, where the correlation and response functions have the time translation invariant form $C(t + \tau, s) = C(\tau)$ and $R(t + \tau, s) = R(\tau)$. Making this substitution, we have

$$\begin{aligned}
0 &= \frac{1}{2}(z + \partial_\tau)C(\tau) + R(-\tau) + \frac{1}{4} \int d\sigma (R(\tau - \sigma)C(\sigma) + C(\tau - \sigma)R(-\sigma)) \\
0 &= \frac{1}{2}(z + \partial_\tau)R(\tau) + \frac{1}{4} \int d\sigma R(\tau - \sigma)R(\sigma) + \frac{1}{2}\delta(\tau)
\end{aligned}$$

Now we can see why using the response function to get the resolvent is easier than using the correlation function: the response function is given by a single closed integral equation! Because it is quadratic, we can solve it with a Fourier transform, which gives

$$0 = \frac{1}{2}(z + i\omega)\hat{R}(\omega) + \frac{1}{4}\hat{R}(\omega)^2 + \frac{1}{2}$$

which is solved to give

$$\hat{R}(\omega) = (z + i\omega) \pm \sqrt{(z + i\omega)^2 - 2} \quad (23)$$

We therefore have

$$G(z) = \hat{R}(0) = z \pm \sqrt{z^2 - 2} \quad (24)$$

again the resolvent of the GOE.

A quick note on the order parameter D : it is always zero in causal theories. This can be seen from the path integral:

$$D(t, s) = \left\langle \sum_{ij} \frac{\delta^2}{\delta h_i(t) \delta h_j(s)} Z[\mathbf{h}] \right\rangle \Big|_{\mathbf{h}=0} \quad (25)$$

The crucial point is that $\langle Z[h] \rangle = 1$ for any choice of $\mathbf{h}$, since there is exactly one trajectory s consistent with each realization of the noise. Therefore, any variation of Z itself is always zero, and likewise D is zero.