11 Isolated eigenvalues and low-rank perturbations

Up to this point in the course we have analyzed the spectral density of random matrices, i.e., the asymptotic probability density function for the eigenvalues of the matrix in the large-N limit. Missing from this analysis are any lonely eigenvalue that are isolated from the rest of the spectrum. When there are only a few such eigenvalues, the asymptotic spectral density is blind to them, since we divide by N. However, such eigenvalues can have an outsized important in the problems where they appear, since their presence can destabilize an otherwise stable equilibrium or encode useful information in a noisy optimization problem.

As an example, consider the matrix formed by adding a rotationally invariant matrix J to a rank-one matrix made from a normalized vector $\boldsymbol{v}$, or

$$H = J + a\boldsymbol{v}\boldsymbol{v}^T \tag{1}$$

When a is very large, we expect that H has an eigenvector very close to $\boldsymbol{v}$ with eigenvalue very close ta a. On the other hand, when a is smaller than the typical eigenvalues of J we expect the perturbation is lost in that sea of eigenvalues. Of interest is the nature of the transition between these regimes.

Consider the resolvent matrix defined by

$$\mathbf{G}(z) = (zI - H)^{-1} = (zI - J - a\boldsymbol{v}\boldsymbol{v}^T)^{-1} \tag{2}$$

We can write this in a more convenient form by making use of a general formula for the inverse matrices perturbed by rank-one perturbations, the Sherman–Morrison formula:

$$(A + \boldsymbol{u}\boldsymbol{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{u}\boldsymbol{v}^T A^{-1}}{1 + \boldsymbol{v}^T A^{-1}\boldsymbol{u}} \tag{3}$$

which can be verified by multiplying $A + \boldsymbol{u}\boldsymbol{v}^T$ on both sides and doing algebra. Applied to our resolvent matrix, we have

$$\mathbf{G}(z) = (zI - J)^{-1} + \frac{a(zI - J)^{-1}\boldsymbol{v}\boldsymbol{v}^T(zI - J)^{-1}}{1 - a\boldsymbol{v}^T(zI - J)^{-1}\boldsymbol{v}} = \mathbf{G}_J(z) + \frac{a\mathbf{G}_J(z)\boldsymbol{v}\boldsymbol{v}^T\mathbf{G}_J(z)}{1 - a\boldsymbol{v}^T\mathbf{G}_J(z)\boldsymbol{v}} \tag{4}$$

where $\mathbf{G}_J(z) = (zI - J)^{-1}$ is the resolvent matrix of the unperturbed matrix J. The location of an isolated eigenvalue of H will correspond to a place where $\mathbf{G}_J(z)$ is nonsingular, because it is outside the spectral density, and where $\mathbf{G}(z)$ has a pole. The conditions for such a pole are that the denominator of the second term vanishes, or

$$0 = 1 - a\boldsymbol{v}^T\mathbf{G}_J(z)\boldsymbol{v} \tag{5}$$

If $\boldsymbol{v}$ is a random vector independent of the eigenvectors of J, then we are safe to write

$$\boldsymbol{v}^T\mathbf{G}_J(z)\boldsymbol{v} = \sum_{i=1}^{N} v_i^2 G_{Jii}(z) \simeq \frac{1}{N}\operatorname{Tr}\mathbf{G}_J(z) = G_J(z) \tag{6}$$

which is simply the resolvent of J. Therefore, we have a pole outside the spectral density when

$$0 = 1 - aG_J(z) \tag{7}$$

or alternatively when

$$z = B_J(a^{-1}) \tag{8}$$

where B_J is the Blue function of J. This gives the location of an isolated eigenvalue due to adding a rank-one perturbation when it exists. It exists when $B_J(a^{-1}$ is outside the edge of the spectral density, and otherwise it does not. Since the Blue function is related to the R transform by

$$R(t) = B(t) - \frac{1}{t} \tag{9}$$

we can alternatively write

$$z = R_J(a^{-1}) + a \tag{10}$$

Recall that the R transform for GOE matrices is

$$R(t) = \sigma^2 t \tag{11}$$

where 2σ is the radius of the spectral density. This means that the continuous part stretches in the region $\pm 2\sigma$. For such matrices, the isolated eigenvalue formula tells us that

$$z = \frac{\sigma^2}{a} + a \tag{12}$$

This is equal to the edge of the spectral density when

$$\pm 2\sigma = \frac{\sigma^2}{a} + a \qquad \longrightarrow \qquad a = \pm\sigma \tag{13}$$

Expanding about a near the transition value gives

$$z = 2\sigma + \frac{(a-\sigma)^2}{\sigma} + O((a-\sigma)^3) \tag{14}$$

which means that the eigenvalue emerges with a quadratic power law from the edge of the spectrum. Interesting, the value of the eigenvalue is greater than a near the transition. The quadratic power is quite general: if $x_\pm$ is the upper or lower edge of the spectral density, then it is a singular point of G_J. In particular, Since $G(x)$ is a decreasing real-valued function for $x > x_+$ or $x < x_-$, this means that

$$\lim_{x \to x_\pm} G_J'(x) = -\infty \tag{15}$$

Since the derivative of G diverges at $x_\pm$, the derivative of its inverse function B vanishes, or

$$\lim_{t\to G_J(x_\pm)} B'_J(t) = 0 \tag{16}$$

The transition point for the isolated pole in the spectral density is given by

$$a^* = \frac{1}{G_J(x_+)} \tag{17}$$

Therefore, expanding about $a = a^*$, we have

$$\begin{aligned} x &= B(\tfrac{1}{a^*}) - \frac{1}{(a^*)^2}B'(\tfrac{1}{a^*})(a-a^*) + \frac{1}{2(a^*)^4}(2a^*B'(\tfrac{1}{a^*}) + B''(\tfrac{1}{a^*}))(a-a^*)^2 + O((a-a^*)^3) \\ &= B(G(x_+)) - \frac{1}{(a^*)^2}B'(G(x_+))(a-a^*) + \frac{1}{2(a^*)^4}(2a^*B'(G(x_+)) + B''(G(x_+)))(a-a^*)^2 + O((a-a^*)^3) \\ &= x_+ + \frac{1}{2(a^*)^4}B''(G(x_+))(a-a^*)^2 + O((a-a^*)^3) \end{aligned}$$

where we have used the fact that $B'(G(x_+)) = 0$ and $B(G(x_+)) = x_+$.

What about the eigenvector associated with the outlier eigenvalue: how well does it reproduce the vector $\boldsymbol{v}$ that was its origin? We can understand the overlap between it and the eigenvector associated with the outlying eigenvalue in the following way. Recall that we can express the resolvent matrix in the following way:

$$\mathbf{G}(z) = \sum_{i=1}^{N} \frac{\boldsymbol{v}_i \boldsymbol{v}_i^T}{z - x_i} \tag{18}$$

for the eigenvectors $\boldsymbol{v}_i$ of H with eigenvalue x_i. Knowing that the outlying eigenvalue will take value $x_o = B_J(a^{-1})$, we can extract the scalar product or overlap by writing

$$\lim_{z\to x_0} (z - x_o)\boldsymbol{v}^T\mathbf{G}(z)\boldsymbol{v} = (\boldsymbol{v}^T\boldsymbol{v}_o)^2 \tag{19}$$

Using the version of the resolvent matrix we wrote earlier, and repeating the argument that $\boldsymbol{v}^T\mathbf{G}_J(z)\boldsymbol{v} \simeq G_J(z)$ if $\boldsymbol{v}$ is free of J, we have

$$\begin{aligned} (\boldsymbol{v}^T\boldsymbol{v}_o)^2 &= \lim_{z\to x_0}(z-x_o)\boldsymbol{v}^T\left(\mathbf{G}_J(z) + \frac{a\mathbf{G}_J(z)\boldsymbol{v}\boldsymbol{v}^T\mathbf{G}_J(z)}{1 - a\boldsymbol{v}^T\mathbf{G}_J(z)\boldsymbol{v}}\right)\boldsymbol{v} \\ &= \lim_{z\to x_0}(z-x_o)\left(G_J(z) + \frac{aG_J(z)^2}{1-aG_J(z)}\right) \\ &= \lim_{z\to x_0}\frac{G_J(z)^2(z-x_o)}{G_J(x_o) - G_J(z)} \end{aligned} \tag{20}$$

where we have used the fact that $a = 1/G_J(x_o)$. To evaluate this limit we must use l'Hospital's rule. Differentiating the top and bottom, we have

$$(\boldsymbol{v}^T\boldsymbol{v}_o)^2 = \lim_{z\to x_0}\frac{G_J(z)^2 + 2G'_J(z)(z-x_o)}{-G'_J(z)} = -\frac{G_J(x_o)^2}{G'_J(x_o)} \tag{21}$$

We can write this in a more compact way using the R transform. By definition,

$$z = B(G(z)) = R(G(z)) + \frac{1}{G(z)} \tag{22}$$

Differentiating both sides, we find

$$1 = R'(G(z))G'(z) - \frac{G'(z)}{G(z)^2} \tag{23}$$

which implies

$$G'(z) = (R'(G(z)) - \tfrac{1}{G(z)^2})^{-1} \tag{24}$$

Therefore,

$$(\boldsymbol{v}^T \boldsymbol{v}_o)^2 = -G_J(x_o)^2 (R'(G(x_o)) - \tfrac{1}{G(x_o)^2}) = 1 - G_J(x_o)^2 R'(G(x_o)) = 1 - a^{-2} R'(a^{-1}) \tag{25}$$

again using $a = G(x_o)^{-1}$. Recall that for GOE matrices $R(t) = \sigma^2 t$. Then $R'(t) = \sigma^2$, and we have

$$(\boldsymbol{v}^T \boldsymbol{v}_o)^2 = 1 - \frac{\sigma^2}{a^2} \tag{26}$$

Recall that the transition value of a for leaving the spectrum is $a^* = \sigma$. Therefore at the transition the overlap begins at zero, then grows steadily to 1 and a goes to infinity, as we would expect. In this case the expansion about the transition is

$$(\boldsymbol{v}^T \boldsymbol{v}_o)^2 = \frac{2}{\sigma}(a - \sigma) + O(a - \sigma)^2 \tag{27}$$

a linear growth. In general the power law for the growth of an overlap at the transition depends on the singularity in the spectral density at the edge.