12 Asymmetric random matrices

We have up to now considered the study of only Hermitian or symmetric random matrices, which have real spectra, are diagonalizable, and have identical left and right eigenvalues. Today we'll examine a standard ensemble of *asymmetric* random matrices, the (real) Ginibre ensemble, which is defined by matrices filled with independent Gaussian real numbers with zero mean and variance $\frac{1}{N}$.

In order to study such ensembles, we will have to change the way that we extract information from the resolvent. When the eigenvalues $\lambda = \lambda_x + i\lambda_y$ are complex, it is still true that the resolvent is related to the spectral density by

$$G(z) = \int d\lambda \, \frac{\rho(\lambda)}{z-\lambda} = \int d\lambda_x \, d\lambda_y \, \frac{\rho(\lambda_x + i\lambda_y)}{z - \lambda_x - i\lambda_y} \tag{1}$$

However, it is no longer necessarily true that a branch cut of G will equal ρ: this is only true when there is a density of exactly real eigenvalues, and such a branch cut only reveals their density, not the density of complex eigenvalues.

Consider integrating G over some closed contour in the complex plane containing a region Ω. On one hand, we have

$$\oint_{\partial\Omega} dz \, G(z) = 2\pi i \frac{1}{N} \overline{\sum_{\lambda \in \Omega} 1} = 2\pi i \int_{\Omega} dz \, \rho(z) \tag{2}$$

Because the contour integral contributes 1 at each pole and that is the number of eigenvalues, the integral is the same as having integrated the spectral density over the region. On the other hand, recall that complex contour integration is the same as a line integral of a two-dimensional vector field, where the real and imaginary parts of G play the role of the vector field. $dz = dx + i dy$ is the infinitesimal line element, and the normal to this is $-dy + i dx$. Using Gauss' law

$$\oint_{\partial\Omega} d\boldsymbol{S} \cdot \boldsymbol{E}(\boldsymbol{x}) = \int_{\Omega} d\boldsymbol{x} \, \boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{x}) \tag{3}$$

we can write

$$\oint_{\partial\Omega} dz \, G(z) = \int_{\Omega} dz \left(-\frac{\partial}{\partial y} + i \frac{\partial}{\partial x} \right) G(z) = 2i \int_{\Omega} dz \, \frac{\partial}{\partial z^*} G(z) \tag{4}$$

where $\frac{\partial}{\partial z^*} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$ is the Wirtinger derivative. Since this is true for every region Ω, we can therefore equate the integrands and write

$$\rho(z) = \frac{1}{\pi} \frac{\partial G(z)}{\partial z^*} \tag{5}$$

We can note a few things here. First, the derivative of any holomorphic function with respect to the conjugate of its argument is always zero. Therefore, there is a nonzero spectral density if and only if the resolvent is nonanalytic. Second, we developed this identity using an analogy with electrostatics, where the real and imaginary part of G

play the role of the electric field. We can extend that analogy by looking for a function $\Phi(z)$ that obeys Poisson's equation

$$\nabla^2\Phi = -\boldsymbol{\nabla}\cdot\boldsymbol{E} = -4\pi\rho \tag{6}$$

or in our case

$$\frac{\partial^2\Phi(z)}{\partial z^*\partial z} = -\frac{\partial G(z)}{\partial z^*} = -\pi\rho(z) \tag{7}$$

This function Φ would represent the scalar potential resulting from 'charges' lying at the eigenvalues. Based on the definition of the resolvent, such a function is easily furnished by writing

$$\Phi(z) = -\frac{1}{N}\overline{\log\det[(z^*I - H^\dagger)(zI-H)]} \tag{8}$$

where we note that

$$\frac{\partial\Phi(z)}{\partial z} = -\frac{1}{N}\overline{\frac{1}{\det[(z^*I-H^\dagger)(zI-H)]}\det[(z^*I-H^\dagger)(zI-H)]\,\mathrm{Tr}[(zI-H)^\dagger(zI-H)]^{-1}(zI-H)^\dagger} = -G(z) \tag{9}$$

as required.

Now we can focus on computing the spectral density for the Ginibre ensemble. We will use the fact that

$$\frac{1}{\det M} = \int\frac{d\boldsymbol{s}}{(\pi)^N}e^{-\boldsymbol{s}^\dagger M\boldsymbol{s}} = \int\frac{d\boldsymbol{s}_x\,d\boldsymbol{s}_y}{(\pi)^N}e^{-\boldsymbol{s}_xM\boldsymbol{s}_x-\boldsymbol{s}_yM\boldsymbol{s}_y} \tag{10}$$

to write

$$\Phi(z) = \frac{1}{N}\overline{\log\int\frac{d\boldsymbol{s}}{(\pi)^N}e^{-\boldsymbol{s}^\dagger(z^*I-H^\dagger)(zI-H)\boldsymbol{s}}} \tag{11}$$

The average over H is blocked by the logarithm. We can bypass it using replicas again, this time using the identity

$$\log x = \lim_{n\to0}x^n\log x = \lim_{n\to0}\frac{\partial}{\partial n}x^n \tag{12}$$

However, this is unnecessary: as for the resolvent of the GOE, we would find the answer is replica symmetric with zero off-diagonal. Alternatively, the determinant is self-averaging. Therefore,

$$\Phi(z) = \frac{1}{N}\log\overline{\int\frac{d\boldsymbol{s}}{(\pi)^N}e^{-\boldsymbol{s}^\dagger(z^*I-H^\dagger)(zI-H)\boldsymbol{s}}} \tag{13}$$

To evaluate this, first define $\boldsymbol{v} = H\boldsymbol{s}$. We can insert $\boldsymbol{v}$ into the integral using δ functions for its real and imaginary parts, with

$$\begin{aligned}\hat{\boldsymbol{v}}_x^T\boldsymbol{v}_x + \hat{\boldsymbol{v}}_y^T\boldsymbol{v}_y &= \frac{1}{2}(\hat{\boldsymbol{v}}_x - i\hat{\boldsymbol{v}}_y)^T(\boldsymbol{v}_x + i\boldsymbol{v}_y) + \frac{1}{2}(\hat{\boldsymbol{v}}_x + i\hat{\boldsymbol{v}}_y)^T(\boldsymbol{v}_x - i\boldsymbol{v}_y)\\ &= \frac{1}{2}\hat{\boldsymbol{v}}^\dagger\boldsymbol{v} + \frac{1}{2}\boldsymbol{v}^\dagger\hat{\boldsymbol{v}}\end{aligned} \tag{14}$$

Therefore we can write

$$\Phi(z) = \frac{1}{N} \log \overline{\int \frac{ds}{(\pi)^N} \frac{dv\, d\hat{v}}{(\pi)^{2N}} e^{-|z|^2 s^\dagger s + (z^* s^\dagger v + z v^\dagger s) - v^\dagger v + i\hat{v}^\dagger (Hs - v) + i(Hs - v)^\dagger \hat{v}}} \quad (15)$$

There is now a linear term in the Gaussian integral over H of the form $i\hat{v}s^\dagger$. Taking the average over H gives

$$\Phi(z) = \frac{1}{N} \log \int \frac{ds}{(\pi)^N} \frac{dv\, d\hat{v}}{(\pi)^{2N}} e^{-|z|^2 s^\dagger s + (z^* s^\dagger v + z v^\dagger s) - v^\dagger v - i\hat{v}^\dagger v - i v^\dagger \hat{v} - \hat{v}^\dagger \hat{v} \frac{1}{N} s^\dagger s} \quad (16)$$

Now, this is a complex Gaussian integral in v and $\hat{v}$ with interaction matrix

$$\begin{bmatrix} I & iI \\ iI & \frac{1}{N} s^\dagger s I \end{bmatrix} \quad (17)$$

and linear term $[z^* s^\dagger, 0]$. By integrating and writing $q = \frac{1}{N} s^\dagger s$, we get

$$\Phi(z) = \frac{1}{N} \log \int \frac{ds}{(\pi)^N} e^{-N|z|^2 q - N \log(q+1) + N|z|^2 \frac{q^2}{q+1}} \quad (18)$$

since

$$\begin{bmatrix} zs \\ 0 \end{bmatrix}^\dagger \begin{bmatrix} I & iI \\ iI & \frac{1}{N} s^\dagger s I \end{bmatrix}^{-1} \begin{bmatrix} zs \\ 0 \end{bmatrix} = N \frac{|z|^2 q^2}{q+1} \quad (19)$$

Making the standard charge of variables from $s^\dagger s$ to q at the cost of $\log q$, we have

$$\Phi(z) = \frac{1}{N} \log \int dq\, e^{-N|z|^2 q - N \log(q+1) + N|z|^2 \frac{q^2}{q+1} + N \log q} \quad (20)$$

$$\Phi(z) = \frac{1}{N} \log \int dq\, e^{-N|z|^2 \frac{q}{q+1} + N \log(\frac{q}{q+1})} \quad (21)$$

Now we are ready to evaluate this by saddle point. Differentiating the effective action gives

$$0 = -\frac{|z|^2}{(1+q)^2} + \frac{1}{q(1+q)} \quad (22)$$

which is solved to give

$$q = \frac{1}{|z|^2 - 1} \quad (23)$$

or $q = \infty$. Note that q cannot be negative, since it in the norm of a complex vector. Therefore, the first solution is only valid for $|z| > 1$. When this is true, we have

$$\Phi(z) = -|z|^2 \frac{q^*}{q^* + 1} + \log(\frac{q^*}{q^* + 1}) = -1 - \log |z|^2 \quad (24)$$

and otherwise we have

$$\Phi(z) = -|z|^2 \frac{q^*}{q^*+1} + \log(\frac{q^*}{q^*+1}) = -|z|^2 \tag{25}$$

Now we can try to find the spectral density. Taking two derivatives, we find when $|z| > 1$,

$$\pi\rho(z) = -\frac{\partial^2 \Phi(z)}{\partial z^* \partial z} = \frac{\partial^2}{\partial z^* \partial z}(1 + \log(z^* z)) = \frac{\partial}{\partial z^*} \frac{z^*}{z^* z} = 0 \tag{26}$$

while when $|z| < 1$,

$$\pi\rho(z) = -\frac{\partial^2 \Phi(z)}{\partial z^* \partial z} = \frac{\partial^2}{\partial z^* \partial z} z^* z = \frac{\partial}{\partial z^*} z^* = 1 \tag{27}$$

Therefore, ρ is constant inside the disk $|z| < 1$ and zero otherwise.

A commonly used extension of this ensemble is to take

$$\overline{H_{ij} H_{kl}} = \frac{1}{N}(\delta_{ik}\delta_{jl} + \tau\delta_{il}\delta_{jk}) \tag{28}$$

When $\tau = 1$, this is precisely the covariance for the GOE ensemble of symmetric Gaussian matrices. Varying τ interpolates the ensemble between symmetric to uncorrelated through matrices whose off-diagonals are asymmetric but correlated. The effect on the spectral density is to warp the disk we described into an ellipse, with a constant nonzero density in the region

$$\frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1 \tag{29}$$

with major and minor axis of radius $1 + \tau$ and $1 - \tau$. Since the area of this ellipse is $\pi(1 - \tau^2)$, the density inside the ellipse is $1/(1 - \tau^2)\pi$. The result of integrating the constant density over just the imaginary numbers is

$$\rho_x(x) = \int dy\, \rho(x+iy) = \frac{1}{(1-\tau^2)\pi} \frac{1-\tau}{1+\tau} \sqrt{(1+\tau)^2 - x^2} = \frac{1}{(1+\tau)^2 \pi} \sqrt{(1+\tau)^2 - x^2} \tag{30}$$

In the limit of $\tau \to 1$, the diameter of minor axis of the ellipse in the imaginary plane shrinks to zero, and the density ρ_x approaches the Wigner semicircle.