Dyson Brownian motion. Recall that the JPDF of eigenvalues for any invariant ensemble of symmetric matrices is given by

$$\rho(\boldsymbol{x}) = \frac{1}{\mathcal{Z}} e^{-\sum_{i=1}^{N} V(x_i) + \sum_{i>j}^{N} \log|x_i - x_j|} \tag{1}$$

where for the GOE the potential is $V(x) = \frac{1}{2}x^2$. We will see how we can sample this distribution using a stochastic process.

1. It is straightforward to sample GOE matrices: just create A with $A_{ij} \sim \mathcal{N}(0,1)$ and write $H = \frac{1}{2}(A + A^T)$. However, we're going to sample them in a more obnoxious way. Consider the stochastic process defined by

   $$\frac{d}{dt} A_{ij}(t) = \xi_{ij}(t) \tag{2}$$

   where ξ is a Gaussian random noise with covariance $\overline{\xi_{ij}(t)\xi_{kl}(t')} = \delta_{ik}\delta_{jl}\delta(t - t')$, e.g., it is white and independent. Write a simulation of this stochastic process, and plot as a function of time the eigenvalues of $H(t) = \frac{1}{2}(A(t) + A(t)^T)$. How do they behave?

   Hint: when a continuous stochastic process is discretized in time with timestep Δt, the variance of the discrete noise should be $2\Delta t \sigma^2$ if σ^2 is the variance of the continuous process. Choose Δt small enough that the eigenvalues do not appear to cross.

2. A result of Freeman Dyson is that the above stochastic process produces an effective stochastic process in the eigenvalues of H of the form

   $$\frac{d}{dt} x_i(t) = \frac{1}{2} \sum_{j \neq i} \frac{1}{x_i - x_j} + \xi_i(t) \tag{3}$$

   where ξ is again a white Gaussian noise with variance one. The 'force' between the eigenvalues means that the stochastic processes are not independent. Show that this force is the derivative of the 'energy' of the Coulomb gas with respect to x_i for $V(x) = 0$. Write a simulation of this stochastic process and compare it with the previous results.

3. The matrices and eigenvalues produced by this process grow unbounded with time. It is therefore convenient to rescale them. What is the distribution of $A_{ij}(t)$ implied by (2) if $A_{ij}(0) \sim \mathcal{N}(0,1)$? If the eigenvalues at time t are rescaled by the standard deviation of $A_{ij}(t)$, do they appear to approach a stationary distribution?

4. How could the deterministic force in the stochastic process (3) be modified to sample arbitrary invariant ensembles? Make such a modification and sample a non-Gaussian ensemble of your choice.

   Hint: make sure the potential is confining!

Jacobian for Hermitian random matrices. Like symmetric matrices, Hermitian ones can be diagonalized by $H = UXU^\dagger$ for a real diagonal matrix X of the eigenvalues and a unitary matrix U of the eigenvectors. Following chapter 7 of the Livan text, show that the determinant of the Jacobian of this transformation is given by

$$\det J(H \to \{\boldsymbol{x}, U\}) = \Delta_N(\boldsymbol{x})^2 = \prod_{i<j}^{N} (x_i - x_j)^2 \tag{4}$$

Hint: the safest way to proceed is to break all complex objects into real and imaginary components and work with real variables throughout. If you want to try a 'complex covariant' approach, remember to treat derivatives with respect to a complex number and its conjugate as independent.