Cavity method for Wishart. Recall that the Wishart ensemble is defined by covariance matrices $W = XX^T$ constructed from $X \in \mathbb{R}^{N\times M}$ whose elements are identically and independently distributed with $X_{i\alpha} \sim \mathcal{N}(0, N^{-1})$. Here we will define their limiting spectral density using the cavity method. The starting point is the block decomposition of W

$$W = \begin{bmatrix} W_{11} & \boldsymbol{W}_1^T \\ \boldsymbol{W}_1 & W_r \end{bmatrix} \tag{1}$$

and the Schur complement formula

$$\frac{1}{(zI - W)^{-1}_{11}} = (z - W_{11}) - \boldsymbol{W}_1^T (zI - W_r)^{-1} \boldsymbol{W}_1 \tag{2}$$

1. Write $\overline{\frac{1}{(zI-W)^{-1}_{11}}}$ in terms of the resolvent. What assumptions are necessary?
2. What is $\overline{z - W_{11}}$? *Hint:* $\overline{W_{11}} \neq 0$
3. Unlike the GOE case, $\boldsymbol{W}_1$ and W_r are not independent. We will need to use some tricks to simplify the last term. Show that the covariance rules $\overline{X_{i\alpha}X_{j\beta}} = \frac{1}{N}\delta_{ij}\delta_{\alpha\beta}$ imply that

   $$\overline{\boldsymbol{W}_1^T (zI - W_r)^{-1} \boldsymbol{W}_1} = \frac{1}{N}\overline{\operatorname{Tr} W_r (zI - W_r)^{-1}} \tag{3}$$

   Hint: Be careful about statistical dependence! Which of the components of X do $\boldsymbol{W}_1$, W_r depend on?
4. We will need one more trick to massage this expression into a form that gives us the resolvent. Show the generic matrix identity

   $$-W_r(zI - W_r)^{-1} = I - z(zI - W_r)^{-1} \tag{4}$$

   What is the result of $\overline{\boldsymbol{W}_1^T (zI - W_r)^{-1} \boldsymbol{W}_1}$?

   Hint: Add $0 = z(zI - W_r)^{-1} - z(zI - W_r)^{-1}$ to the lefthand side.
5. Solve for the resolvent and use it to calculate the spectral density.

Replicas for Wishart. Recall that the replica method starts from the identity

$$G(z) = \lim_{n\to 0} \int \left(\prod_{a=1}^n d\boldsymbol{s}_a\right) \frac{\boldsymbol{s}_1^T\boldsymbol{s}_1}{N} \overline{e^{-\frac{1}{2}\sum_{a=1}^n \boldsymbol{s}_a^T (zI-W)\boldsymbol{s}_a}} \tag{5}$$

We cannot average over the matrix ensemble directly because W is not Gaussian. Instead we define temporary variables $\boldsymbol{v}_a \in \mathbb{R}^M$ defined by $\boldsymbol{v}_a = X^T\boldsymbol{s}_a$ for each replica $a = 1, \ldots, n$. Using a δ function to enforce this definition and exponentiating it gives

$$\begin{aligned} G(z) &= \lim_{n\to 0} \int \left(\prod_{a=1}^n d\boldsymbol{s}_a\right) \frac{\boldsymbol{s}_1^T\boldsymbol{s}_1}{N} \overline{e^{-\frac{1}{2}\sum_{a=1}^n (z\boldsymbol{s}_a^T\boldsymbol{s}_a - \boldsymbol{s}_a^T XX^T\boldsymbol{s}_a)}} \\ &= \lim_{n\to 0} \int \left(\prod_{a=1}^n d\boldsymbol{s}_a\, d\boldsymbol{v}_a\right) \frac{\boldsymbol{s}_1^T\boldsymbol{s}_1}{N} \overline{e^{-\frac{1}{2}\sum_{a=1}^n (z\boldsymbol{s}_a^T\boldsymbol{s}_a - \boldsymbol{v}_a^T\boldsymbol{v}_a)} \prod_{a=1}^n \delta(\boldsymbol{v}_a - X^T\boldsymbol{s}_a)} \\ &= \lim_{n\to 0} \int \left(\prod_{a=1}^n d\boldsymbol{s}_a\, d\boldsymbol{v}_a\, \frac{d\hat{\boldsymbol{v}}_a}{(2\pi)^M}\right) \frac{\boldsymbol{s}_1^T\boldsymbol{s}_1}{N} \overline{e^{-\frac{1}{2}\sum_{a=1}^n (z\boldsymbol{s}_a^T\boldsymbol{s}_a - \boldsymbol{v}_a^T\boldsymbol{v}_a) + \sum_{a=1}^n i\hat{\boldsymbol{v}}_a^T(\boldsymbol{v}_a - X^T\boldsymbol{s}_a)}} \end{aligned} \tag{6}$$

which is an exponential linear in X and therefore is ripe to be averaged. Make the average over X and follow the replica method as before to rederive the Wishart spectral density.

Hint: After averaging over X, you will find a Gaussian integral in $\begin{bmatrix} \boldsymbol{v}_a \\ \hat{\boldsymbol{v}}_a \end{bmatrix}$, so the $\boldsymbol{v}_a$ and $\hat{\boldsymbol{v}}_a$ can be integrated away!