Factorization of free products. If X and Y are free, then

$$\tau((X^{n_1} - \tau(X^{n_1}))(Y^{m_1} - \tau(Y^{m_1}))(X^{n_2} - \tau(X^{n_2}))(Y^{m_2} - \tau(Y^{m_2}))\cdots) = 0 \tag{1}$$

for any set of integers $n_1, m_1, n_2, m_2, \ldots$. Use this property to prove that

$$\tau(X^2Y^2) = \tau(X^2)\tau(Y^2) \tag{2}$$

$$\tau(XYXY) = \tau(X)^2\tau(Y^2) + \tau(X^2)\tau(Y)^2 - \tau(X^2)\tau(Y^2) \tag{3}$$

***R* transform for Wishart.** Recall that the resolvent of the Wishart ensemble is given by the equation

$$\frac{1}{G_W(z)} = z - \alpha + 1 - zG_W(z) \tag{4}$$

where $\alpha = M/N$ is the ratio of dimensions of the rectangular matrix $X = \mathbb{R}^{N\times M}$, with $\overline{X_{ia}X_{jb}} = \frac{1}{N}\delta_{ij}\delta_{ab}$ and $W = XX^T$.

1. Calculate the R transform of W.
2. Use the R transform you calculated to compute the R transform of the weighted sum of a GOE matrix H and Wishart matrix, with $M = H + aW$ for some constant a.
3. Invert the R transform to find the resolvant and spectral density of M.
4. Generate samples of M and compare the histogram of eigenvalues to your prediction for a selection of a.