1 Mathematical prerequisites

Gaussian integration. Multidimensional Gaussian integrals are important tools of the trade for theoretical physicists.

1. Making use of polar coordinates, show that

$$\left(\int_{\mathbb{R}} dx\, e^{-\frac{1}{2}x^2}\right)^2 = \int_{\mathbb{R}} dx\, e^{-\frac{1}{2}x^2} \int_{\mathbb{R}} dy\, e^{-\frac{1}{2}y^2} = 2\pi \tag{1}$$

2. By making a change of variables, show that

$$\int_{\mathbb{R}} dx\, e^{-\frac{1}{2}ax^2+bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}a^{-1}b^2} \tag{2}$$

3. Let A be an $N \times N$ real symmetric matrix and $\mathbf{b} \in \mathbb{R}^N$. Making a change of variables that diagonalizes A, show that

$$\int_{\mathbb{R}^N} d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}+\mathbf{b}\cdot\mathbf{x}} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}\mathbf{b}^T A^{-1}\mathbf{b}} \tag{3}$$

The Dirac δ function. We will make extensive use of the Fourier representation of the Dirac δ function. Consider its Gaussian regularization

$$\delta(x) = \lim_{\epsilon\to 0} \delta_\epsilon(x) \qquad \delta_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2}\epsilon^{-1}x^2} \tag{4}$$

1. Show that

$$\int_{\mathbb{R}} dx\, \delta_\epsilon(x) = 1 \tag{5}$$

2. Show that the Fourier transform of $\delta_\epsilon(x)$ is given by

$$\hat{\delta}_\epsilon(\hat{x}) \equiv \int_{\mathbb{R}} dx\, e^{-i\hat{x}x}\delta_\epsilon(x) = e^{-\frac{1}{2}\epsilon\hat{x}^2} \tag{6}$$

3. Taking the inverse Fourier transform and the limit, show that

$$\delta(x) = \int \frac{d\hat{x}}{2\pi}\, e^{i\hat{x}x} \tag{7}$$

Wick's theorem. We will sometimes use a specific formulation of Wick's theorem.

1. If x is a Gaussian random variable with zero mean, use integration by parts to show that

$$\langle x f(x)\rangle = \langle x^2\rangle\langle f'(x)\rangle \tag{8}$$

Hint: the PDF of Gaussian x is $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$ for $\sigma^2 = \langle x^2\rangle$.

2. If $x \in \mathbb{R}^N$ is a vector of Gaussian random variables with zero mean (not necessarily independent), show that

$$\langle x_1 \cdot f(x) \rangle = \sum_{i=1}^{N} \langle x_1 x_i \rangle \left\langle \frac{\partial f}{\partial x_i}(x) \right\rangle \tag{9}$$

Hint: the PDF of Gaussian x is $p(x) = \frac{1}{\sqrt{(2\pi)^N \det \Sigma}} e^{-\frac{1}{2} x^T \Sigma^{-1} x}$ where $\Sigma_{ij} = \langle x_i x_j \rangle$ is the *covariance matrix* of x.

Saddle-point approximation (real). We will often encounter integrals of the form

$$\mathcal{I} = \int_{\mathbb{R}^d} dx \, e^{N\mathcal{S}(x)} \tag{10}$$

for N a very large number. We can approximate the result of such integrals by their value only at maxima of $\mathcal{S}$.

1. Suppose x^* is a maximum of S. Taylor expanding S about x^*, show that the integrand can be written

$$e^{N\mathcal{S}(x)} \simeq e^{N\mathcal{S}(x^*)} e^{\frac{N}{2}(x - x^*)^T \mathcal{S}''(x^*)(x - x^*)} \tag{11}$$

where $\mathcal{S}''(x^*)$ is the Hessian matrix of second derivatives of $\mathcal{S}$.

2. Using this approximation of the integrand, show that

$$\mathcal{I} \simeq \sum_{\sigma} e^{N\mathcal{S}(x^*_\sigma)} \sqrt{\frac{(2\pi/N)^d}{\det(-\mathcal{S}''(x^*_\sigma))}} \tag{12}$$

where the sum is over maxima of $\mathcal{S}$. Argue that the determinant of the negative Hessian is always positive, making this expression well-defined.

3. Suppose that x^* is the global maximum of $\mathcal{S}$. Argue that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{I} = \mathcal{S}(x^*) \tag{13}$$

Sterling's formula. A nice application of the saddle-point approximation is to quickly derive Stirling's formula. The factorial coincides with the Γ function when evaluated at natural numbers, with

$$N! = \Gamma(N + 1) = \int_0^\infty dt \, t^N e^{-t} \tag{14}$$

Change variables to $u = N^{-1}t$ and manipulate the integral to bring it to the form (10), then evaluate it using the saddle-point approximation for large N.

Saddle-point approximation (complex). Why is it called the saddle-point approximation and not the maximum approximation? Because *any* stationary point of $\mathcal{S}$ that maximizes $\operatorname{Re}\mathcal{S}$ should be summed over, whether or not it is in the domain of integration. Saddle points of $\mathcal{S}$ in the complex plane should also be considered!

1. If $\mathcal{S}$ is an analytic function, then we can deform the contour of integration $\mathbb{R}^d$ at will. It is convenient to pick a contour where $\operatorname{Im}\mathcal{S}$ is piecewise constant to avoid oscillations. Show that $\operatorname{Im}\mathcal{S}$ is constant under gradient ascent on $\operatorname{Re}\mathcal{S}$ defined for $z = x + iy$ by

$$\dot{z} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\operatorname{Re}\mathcal{S} \tag{15}$$

   Hint: Differentiate $\operatorname{Im}\mathcal{S}$ with time and use the chain rule. If $\mathcal{S}$ is analytic it satisfies the Cauchy–Riemann equations

$$\frac{\partial \operatorname{Re}\mathcal{S}}{dx} = \frac{\partial \operatorname{Im}\mathcal{S}}{dy} \qquad \frac{\partial \operatorname{Re}\mathcal{S}}{dy} = -\frac{\partial \operatorname{Im}\mathcal{S}}{dx} \tag{16}$$

2. The integral is only well defined if the real part of the action is bounded from above, and we cannot deform the contour so that it diverges. Therefore, contours defined by gradient ascent must terminate at maxima. Around each maximum z^*, we establish a set of coordinates u in which

$$\mathcal{S}(z(u)) = \mathcal{S}(z^*) - \frac{1}{2}u^T u \tag{17}$$

   Show that the determinant of the Jacobian of the coordinate transformation at the maximum can be written

$$\det z'(0) = [\det(-\mathcal{S}''(z^*))]^{-\frac{1}{2}} \tag{18}$$

   Hint: Expand $\mathcal{S}(z(u))$ in u and require that (17) is satisfied.

3. When written in the coordinates u, the action has constant imaginary part for $u \in \mathbb{R}^d$. Therefore, by moving the contour of integration to the constant-imaginary-part surfaces attached to maxima of $\operatorname{Re}\mathcal{S}$ and making a change of variables on each one from z to u, we can write our integral as

$$\mathcal{I} = \sum_\sigma e^{N\mathcal{S}(z_\sigma^*)} \int_{\mathbb{R}^d} du \, |\det z_\sigma'(u)| \, e^{-\frac{1}{2}Nu^T u} \tag{19}$$

   where the sum is over maxima z_σ^* of $\operatorname{Re}\mathcal{S}$. Approximate each integral by treating $e^{-\frac{1}{2}Nu^2} = \sqrt{\frac{2\pi}{N}}\delta_{N^{-1}}(u) \simeq \sqrt{\frac{2\pi}{N}}\delta(u)$, and show that

$$\mathcal{I} \simeq \sum_\sigma e^{N\mathcal{S}(z_\sigma^*)} \sqrt{\frac{(2\pi/N)^d}{\det(-\mathcal{S}''(z_\sigma^*))}} \tag{20}$$