2 Equilibrium statistical mechanics

Binary spin Curie–Weiss model. The traditional Curie–Weiss model uses binary spins, not spherical ones. Here we will work out its mean-field theory. The partition function is

$$Z = \sum_{s=\{\pm 1\}^N} e^{\frac{1}{2}\beta \frac{1}{N} \sum_{ij=1}^N s_i s_j} \tag{1}$$

1. Define the average magnetization $m = \frac{1}{N}\sum_{i=1}^N s_i$. Argue that we can write Z as

$$Z = \sum_{m=-1}^{1} e^{\frac{1}{2}\beta N m^2} \sum_{s=\{\pm 1\}^N} \delta_{Nm - \sum_{i=1}^N s_i} \tag{2}$$

   where the sum over m has spacings $\Delta m = \frac{1}{N}$.

2. Evaluating the entropy is now a combinatorial problem. Argue that

$$\sum_{s=\{\pm 1\}^N} \delta_{Nm - \sum_{i=1}^N s_i} = \frac{N!}{(\frac{1}{2}N(1+m))!(\frac{1}{2}N(1-m))!} \tag{3}$$

   Hint: For a given magnetization m, how many spins are up? How many ways are there to choose the up spins from the set of all spins?

3. Use Sterling's formula

$$M! \simeq \sqrt{2\pi M}\left(\frac{M}{e}\right)^M = \sqrt{2\pi M} e^{M(\log M - 1)} \tag{4}$$

   to show that

$$\sum_{s=\{\pm 1\}^N} \delta_{Nm - \sum_{i=1}^N s_i} \simeq \sqrt{\frac{2}{\pi N(1-m^2)}} e^{N\log 2 - N\int_0^m dm' \tanh^{-1}(m')} \tag{5}$$

   Hint: $\tanh^{-1}(x) = \frac{1}{2}\log\frac{1+x}{1-x} = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$, and $\int_0^m dm' \tanh^{-1}(m') = m\tanh^{-1}(m) + \frac{1}{2}\log(1-m^2)$.

4. Putting the entropy together with the energy gives

$$Z \simeq \int dm \sqrt{\frac{2}{\pi N(1-m^2)}} e^{N\mathcal{S}(m)} \tag{6}$$

   for effective action

$$\mathcal{S}(m) = \log 2 + \frac{1}{2}\beta m^2 - \int_0^m dm' \tanh^{-1}(m') \tag{7}$$

Show that $F = -\frac{1}{\beta}\mathcal{S}(m^*)$ where

$$m^* = \begin{cases} 0 & \beta < 1 \\ m^*_\pm & \beta > 1 \end{cases} \tag{8}$$

where $m^*_\pm$ are nonzero solutions to $m^*_\pm = \tanh(\beta m^*_\pm)$.

Hint: $\tanh'(0) = 1$, $\tanh''(0) = 0$, $\tanh'''(0) = -2$.

5. In the spherical Curie–Weiss model, we saw that $m^* \propto |T - T_c|^{\frac{1}{2}}$. By expanding about small m^* near the critical point, show that the binary Curie–Weiss model has the same scaling.

Entropy in binary and continuous systems. One qualitative difference between binary and continuous systems is the zero-temperature entropy. Recall that the entropy can be derived from the free energy by

$$S = \frac{1}{T}(E - F) = \beta\left(\frac{\partial(\beta F)}{\partial\beta} - F\right) = \beta^2\frac{\partial F}{\partial\beta} \tag{9}$$

1. We showed that the free energy per site of the spherical Curie–Weiss model in the low-temperature phase is

$$F = -\frac{1}{2\beta}(1 + \log(2\pi)) - \frac{1}{2}\frac{\beta - 1}{\beta} + \frac{1}{2\beta}\log\beta \tag{10}$$

Show that the entropy of the model goes to $-\infty$ at zero temperature.

2. We don't have an explicit formula for the free energy per site of the binary Curie–Weiss model at low temperature. However, we can still work out the entropy per site in the zero-temperature limit. At very low temperatures, $m^* \simeq 1 - \Delta m$ for small Δm. Show that

$$\Delta m \simeq 1 - \tanh\beta \qquad \text{and therefore} \qquad m \simeq \tanh\beta \tag{11}$$

Hint: Expand the relation $m^* = \tanh(\beta m^*)$ and solve. $\tanh'(x) = \cosh(x)^{-2}$. Keep only the largest terms at large β.

3. Show that for large β, the free energy per site is

$$F \simeq -\frac{1}{\beta}\left(\frac{1}{2}\beta(\tanh\beta)^2 - \beta\tanh\beta + \beta\right) \tag{12}$$

Hint: $1 - (\tanh\beta)^2 = (\cosh\beta)^{-2} \simeq 4e^{-2\beta}$

4. Show that the entropy of the model goes to 0 at zero temperature.

5. Why are the two cases different? Explain why entropy in the continuous model approaches $-\infty$ and entropy in the binary model approaches 0.

Susceptibilities. The magnetic susceptibility is a common way to characterize phase transitions. It is defined by adding an external field h to the model, with $H(s \mid h) = H(\mathbf{s}) - h\sum_i s_i = H(\mathbf{s}) - Nhm(\mathbf{s})$. Then the susceptibility is

$$\chi = \left.\frac{\partial\langle m\rangle}{\partial h}\right|_{h=0} \tag{13}$$

1. Show that in equilibrium,
$$\chi = \beta N(\langle m^2\rangle - \langle m\rangle^2) \tag{14}$$

2. We calculate χ using the mean-field effective action. Argue that
$$\langle m^2\rangle = \frac{\int dm\, e^{N\mathcal{S}(m)} m^2}{\int dm\, e^{N\mathcal{S}(m)}} \tag{15}$$

3. If $\pm m^*$ maximizes $\mathcal{S}$, argue that
$$\langle m^2\rangle - \langle m\rangle^2 = \frac{\int dm\,(m-m^*)^2 e^{\frac{1}{2}N\mathcal{S}''(m^*)(m-m^*)^2}}{\int dm\, e^{\frac{1}{2}N\mathcal{S}''(m^*)(m-m^*)^2}} = -\frac{1}{N\mathcal{S}''(m^*)} \tag{16}$$
Therefore susceptibilities can be extracted directly from the curvature of the effective action at the saddle point.

4. Show for the Curie–Weiss model that the susceptibility diverges at the phase transition like $|T - T_c|^{-1}$ (you can use the spherical or the binary).
Hint: It is easier to show this in the high temperature phase!

Susceptibility in the antiferromagnet. Susceptibilities only diverge if the field couples directly to the order parameter. In antiferromagnets, the order parameter is not the magnetization. Instead, the spins are split into two equal groups (sublattices) with separate magnetizations m_1 and m_2, and the antiferromagnetic order parameter is the difference $m_a = m_1 - m_2$ between the magnetizations of the two groups. The Hamiltonian for the mean-field antiferromagnet is $-\frac{1}{2}Nm_a^2$.

1. Argue that the effective action for the binary antiferromagnet in terms of m_a and $m = m_1 + m_2$ is given by
$$\mathcal{S}(m_a, m) = \frac{1}{2}\beta m_a^2 + \log 2 - \frac{1}{2}\int_0^{m+m_a} dm'\, \tanh^{-1}(m') - \frac{1}{2}\int_0^{m-m_a} dm'\, \tanh^{-1}(m')$$

2. If $m = 0$, argue that $\mathcal{S}(m_a, 0)$ is the same as the effective action for the Curie–Weiss model. Therefore, the two have identical critical behavior.

3. Show that
$$\left.\frac{\partial^2\mathcal{S}}{\partial^2 m}\right|_{m=0} = -\frac{1}{1-m_a^2} \tag{17}$$
What is the behavior of the susceptibility at the critical point?
Hint: $\frac{\partial}{\partial x}\tanh^{-1}(x) = (1-x^2)^{-1}$