3 Annealed disordered systems & Langevin dynamics

Matrix calculus I. We will make use of several useful properties of matrices.

1. If A is an $N \times N$ matrix, use the Gaussian integral identity

$$\det A = (2\pi)^N \left(\int ds\, e^{-\frac{1}{2} s^T A s} \right)^{-2} \tag{1}$$

and the chain rule to show that

$$\frac{\partial}{\partial A_{ij}} \det A = \det A \, A^{-1}_{ij} \tag{2}$$

Hint: $\langle s_i s_j \rangle = \Sigma_{ij}$ if $\mathbf{s}$ is Gaussian with covariance matrix Σ.

Gaussian averages. We often find an exponential containing a centered (zero-mean) Gaussian random variable. Show that if $\mathbf{s} \in \mathbb{R}^N$ is centered Gaussian with generic covariance $\overline{s_i s_j}$ and $\mathbf{b} \in \mathbb{R}^N$ is a constant vector, then

$$\overline{e^{\mathbf{b}\cdot\mathbf{s}}} = e^{\frac{1}{2}\sum_{ij} b_i b_j \overline{s_i s_j}} \tag{3}$$

Annealed Sherrington–Kirkpatrick model. What does the annealed calculation look like for binary spins? The Sherrington–Kirkpatrick model is

$$H_J(\mathbf{s}) = -\frac{1}{2\sqrt{N}} \sum_{ij} J_{ij} s_i s_j \tag{4}$$

for binary spins $\mathbf{s} = \{\pm 1\}^N$ and for Gaussian $\overline{J_{ij}} = 0$, $\overline{J_{ij} J_{kl}} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. Show that the annealed partition function is

$$\overline{Z_J} = \overline{\sum_{\mathbf{s}\in\{\pm 1\}^N} e^{-\beta H_J(\mathbf{s})}} = e^{N \log 2 + \frac{1}{4} N \beta^2} \tag{5}$$

with exactly the same β-dependence as the spherical case!

From Langevin to Boltzmann. We introduced dynamics using the Langevin equation

$$\dot{\mathbf{s}} = -\boldsymbol{\nabla} H(\mathbf{s}) + \boldsymbol{\xi} \tag{6}$$

where $\boldsymbol{\xi}$ is a centered Gaussian noise with variance $\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t - t')$. In this problem we will show that this dynamics implies a Boltzmann distribution when it is stationary.

1. After a small timestep Δt, the change in $\mathbf{s}$ will be

$$\Delta\mathbf{s} = -\boldsymbol{\nabla} H(\mathbf{s})\Delta t + \int_0^{\Delta t} dt\, \boldsymbol{\xi}(t) \tag{7}$$

   Argue that the distribution of steps $p(\Delta\mathbf{s} \mid \mathbf{s})$ starting from $\mathbf{s}$ is Gaussian with mean $-\boldsymbol{\nabla} H(\mathbf{s})\,\Delta t$ and covariance $\langle(\Delta s_i - \langle \Delta s_i\rangle)(\Delta s_j - \langle \Delta s_i\rangle)\rangle = 2T\delta_{ij}\,\Delta t$.

2. Consider an arbitrary function $f : \mathbb{R}^N \to \mathbb{R}$. Argue that

$$f(\mathbf{s}(t+\Delta t)) = f(\mathbf{s}(t)) + \boldsymbol{\nabla} f(\mathbf{s}(t))\cdot\Delta\mathbf{s} + \frac{1}{2}\Delta\mathbf{s}^T\boldsymbol{\nabla}^2 f(\mathbf{s}(t))\Delta\mathbf{s} + O(\Delta\mathbf{s}^3) \tag{8}$$

   Taking the average over the distribution of steps, show that

$$\frac{\partial}{\partial t}\langle f(\mathbf{s}(t))\rangle = \langle -\boldsymbol{\nabla} f(\mathbf{s}(t))\cdot\boldsymbol{\nabla} H(\mathbf{s}(t)) + T(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}) f(\mathbf{s}(t))\rangle \tag{9}$$

3. If the dynamics have reached a stationary state, then $\frac{\partial}{\partial t}\langle f(\mathbf{s}(t))\rangle = 0$ for any function f. If $p(\mathbf{s})$ is the probability distribution of $\mathbf{s}$ in the stationary state, we can therefore conclude

$$\begin{aligned} 0 &= \langle -\boldsymbol{\nabla} f(\mathbf{s}(t))\cdot\boldsymbol{\nabla} H(\mathbf{s}(t)) + T(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}) f(\mathbf{s}(t))\rangle \\ &= \int d\mathbf{s}\, p(\mathbf{s})\,[-\boldsymbol{\nabla} f(\mathbf{s})\cdot\boldsymbol{\nabla} H(\mathbf{s}) + T(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}) f(\mathbf{s})] \end{aligned} \tag{10}$$

   Making an integration by parts, show that this implies

$$0 = \boldsymbol{\nabla} H(\mathbf{s}) p(\mathbf{s}) + T\boldsymbol{\nabla} p(\mathbf{s}) \tag{11}$$

4. Show that the differential equation above is solved by $p(\mathbf{s}) \propto e^{-\beta H(\mathbf{s})}$.