

4 Replica symmetry

Matrix calculus II. We will make use of several useful properties of matrices.

1. Use the identity

$$A_{ij}^{-1} = \langle s_i s_j \rangle = \sqrt{\frac{\det A}{(2\pi)^N}} \int ds e^{-\frac{1}{2} s^T A s} s_i s_j \quad (1)$$

and Wick's theorem to show that

$$\frac{\partial A_{ij}^{-1}}{\partial A_{kl}} = -\frac{1}{2} A_{ik}^{-1} A_{jl}^{-1} - \frac{1}{2} A_{il}^{-1} A_{jk}^{-1} \quad (2)$$

Hint: Wick's theorem is $\langle s_i s_j s_k s_l \rangle = \langle s_i s_j \rangle \langle s_k s_l \rangle + \langle s_i s_k \rangle \langle s_j s_l \rangle + \langle s_i s_l \rangle \langle s_j s_k \rangle$.

Replica symmetry in the Sherrington–Kirkpatrick model. The spherical model is one of the only disordered systems model where the integral \mathbf{s} can be made for arbitrary Q . In most systems, including binary ones, the integral can only be made *after* we have chosen a specific structure for Q . We will see how this works for replica symmetric Q in the Sherrington–Kirkpatrick model, with Hamiltonian

$$H_J(\mathbf{s}) = -\frac{1}{2\sqrt{N}} \sum_{ij} J_{ij} s_i s_j \quad (3)$$

for Gaussian $\overline{J_{ij}} = 0$, $\overline{J_{ij} J_{kl}} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}$.

1. Show that the average replicated partition function can be written

$$\overline{Z_J^n} = \sum_{\mathbf{s}_1 \in \{\pm 1\}^N} \dots \sum_{\mathbf{s}_n \in \{\pm 1\}^N} e^{\frac{1}{4} \frac{1}{N} \beta^2 \sum_{ab}^n (\mathbf{s}_a \cdot \mathbf{s}_b)^2} \quad (4)$$

2. Introduce the order parameter matrix $Q_{ab} = \frac{1}{N} \mathbf{s}_a \cdot \mathbf{s}_b$ of overlaps. By inserting a *Dirac* δ -function to fix the definition of Q and using its Fourier representation with auxiliary variable \tilde{Q} , show that the average replicated partition function can be written

$$\overline{Z_J^n} = \int dQ d\tilde{Q} e^{\frac{1}{4} N \beta^2 \sum_{ab}^n Q_{ab}^2 - \frac{1}{2} N \sum_{ab}^n \tilde{Q}_{ab} Q_{ab}} \sum_{\mathbf{s}_1 \in \{\pm 1\}^N} \dots \sum_{\mathbf{s}_n \in \{\pm 1\}^N} e^{\frac{1}{2} \sum_{ab} \tilde{Q}_{ab} \mathbf{s}_a \cdot \mathbf{s}_b}$$

3. In many models, this point is as far as we can go without making an ansatz for Q and \tilde{Q} . However, in *this* model we can eliminate one by saddle point. Argue that $\tilde{Q} = \beta^2 Q$ and

$$Z_J^n = \int dQ e^{-\frac{1}{4} N \beta^2 \sum_{ab}^n Q_{ab}^2} \sum_{\mathbf{s}_1 \in \{\pm 1\}^N} \dots \sum_{\mathbf{s}_n \in \{\pm 1\}^N} e^{\frac{1}{2} \beta^2 \sum_{ab} Q_{ab} \mathbf{s}_a \cdot \mathbf{s}_b} \quad (5)$$

4. To move further we must make an ansatz for \tilde{Q} . Suppose it is replica symmetric, with $Q_{ab} = \delta_{ab} + (1 - \delta_{ab})q$. Show that

$$e^{-\frac{1}{4}N\beta^2 \sum_{ab}^n Q_{ab}^2} = e^{-\frac{1}{4}Nn\beta^2(1+(n-1)q^2)} \quad (6)$$

$$e^{\frac{1}{2}\beta^2 \sum_{ab}^n Q_{ab} s_a \cdot s_b} = e^{\frac{1}{2}Nn\beta^2(1-q)} e^{\frac{1}{2}\beta^2 q \sum_{ab}^n s_a \cdot s_b} \quad (7)$$

5. To make the sum over the s_a , we need to use one more trick, called a *Hubbard–Stratonovich transformation*. You may know it better as Gaussian integration. Argue that

$$e^{\frac{1}{2}\beta^2 q \sum_{ab}^n s_a \cdot s_b} = \prod_{i=1}^N \int \frac{dz_i}{\sqrt{2\pi\beta^2 q}} e^{-\frac{1}{2}(\beta^2 q)^{-1}z_i^2 + z_i \sum_a^n s_{ai}} \quad (8)$$

6. By making the Hubbard–Stratonovich transformation, we have decoupled the components s_{ai} in a way that factorizes. Use this to show that

$$\sum_{s_1 \in \{\pm 1\}^N} \dots \sum_{s_n \in \{\pm 1\}^N} e^{\frac{1}{2}\beta^2 q \sum_{ab}^n s_a \cdot s_b} = \left(\int \frac{dz}{\sqrt{2\pi\beta^2 q}} e^{-\frac{1}{2}(\beta^2 q)^{-1}z^2} (2 \cosh z)^n \right)^N$$

7. Combining the pieces worked out above, the average replicated partition function is

$$\overline{Z_1^n} = \int dq e^{NnS_n^{\text{RS}}(q)} \quad (9)$$

where the effective action is

$$S_n^{\text{RS}}(q) = -\frac{1}{4}\beta^2(1+(n-1)q^2) + \frac{1}{2}\beta^2(1-q) + \frac{1}{n} \log \left[\int \frac{dz}{\sqrt{2\pi\beta^2 q}} e^{-\frac{1}{2}(\beta^2 q)^{-1}z^2} (2 \cosh z)^n \right] \quad (10)$$

Argue that the limit of n to zero gives

$$S^{\text{RS}}(q) = \lim_{n \rightarrow 0} S_n^{\text{RS}}(q) = \frac{1}{4}\beta^2 + \frac{1}{2}\beta^2(\frac{1}{2}q^2 - q) + \int \frac{dz}{\sqrt{2\pi\beta^2 q}} e^{-\frac{1}{2}(\beta^2 q)^{-1}z^2} \log(2 \cosh z) \quad (11)$$

Hint: What is $\int \frac{dx}{\sqrt{2\pi a}} e^{-\frac{1}{2}a^{-1}x^2}$? You may need L'Hôpital's rule...

8. Argue that the saddle point condition for q is

$$0 = q - 1 + \int \frac{dz}{\sqrt{2\pi\beta^2 q}} e^{-\frac{1}{2}(\beta^2 q)^{-1}z^2} (\cosh z)^{-2} \quad (12)$$

Hint: You can use $\cosh' x = \sinh x$, $\tanh' x = (\cosh x)^{-2}$, and the useful Gaussian identity $\frac{\partial}{\partial a} \int \frac{dx}{\sqrt{2\pi a}} e^{-\frac{1}{2}a^{-1}x^2} f(x) = \frac{1}{2} \int \frac{dx}{\sqrt{2\pi a}} e^{-\frac{1}{2}a^{-1}x^2} f''(x)$ (provable using $\frac{\partial}{\partial a} \frac{e^{-\frac{1}{2}a^{-1}x^2}}{\sqrt{2\pi a}} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \frac{e^{-\frac{1}{2}a^{-1}x^2}}{\sqrt{2\pi a}}$ and integration by parts)

9. Argue that $q = 0$ is always a solution. What is $\overline{\log Z}$ in this case?

Hint: What is $\lim_{\alpha \rightarrow 0} \frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{1}{2}\alpha^{-1}x^2}$?

10. The phase transition in this model is continuous, so we can find the critical temperature by expanding about small q . Argue that

$$0 = q - \beta^2 q + 2\beta^4 q^2 + O(q^4) \quad (13)$$

and therefore there is another solution for $\beta > 1$.

Hint: Use hints above! $\frac{\partial^2}{\partial z^2}(\cosh z)^{-2}|_{z=0} = -2$, $\frac{\partial^4}{\partial z^4}(\cosh z)^{-2}|_{z=0} = 16$

11. At very low temperature, q approaches 1 (why?). Therefore, we can write $q \simeq 1 - y$ for small y . Show that

$$0 = -y + \sqrt{\frac{2}{\pi}}\beta^{-1} + O(\beta^{-2}) \quad (14)$$

and therefore that

$$\frac{1}{N} \overline{\log Z} = \frac{1}{2\pi} + O(\beta^{-1}) \quad (15)$$

Hint: What is $\lim_{\alpha \rightarrow \infty} e^{-\frac{1}{2}\alpha^{-1}x^2}$? $\int_{-\infty}^{\infty} dx (\cosh x)^{-2} = 2$.

12. Show that the entropy of the model at zero temperature is $-\frac{N}{2\pi}$. Why is this bad?