5 Replica symmetry breaking

1RSB in the Sherrington–Kirkpatrick model. Last week you showed that the replicated partition function for the SK model is

$$\overline{Z_J^n} = \int dQ\, e^{-\frac{1}{4}N\beta^2\sum_{ab}^n Q_{ab}^2} \sum_{s_1\in\{\pm1\}^N} \cdots \sum_{s_n\in\{\pm1\}^N} e^{\frac{1}{2}\beta^2\sum_{ab} Q_{ab} s_a\cdot s_b} \tag{1}$$

for general matrix Q. Today we will see how to write the effective action for Q with 1RSB structure. Convince yourself 1RSB Q can be written

$$Q_{ab} = (1-q_1)\delta_{ab} + (q_1-q_0)\delta_{\lceil a/m\rceil,\lceil b/m\rceil} + q_0 \tag{2}$$

where $\lceil x\rceil$ is notation for the *ceiling* function, which rounds the expression to the nearest integer above.

1. Argue that for 1RSB Q, the two terms in the action can be written

$$e^{-\frac{1}{4}N\beta^2\sum_{ab}^n Q_{ab}^2} = e^{-\frac{1}{4}Nn\beta^2(1+(m-1)q_1^2+(n-m)q_0^2)} \tag{3}$$

$$e^{\frac{1}{2}\beta^2\sum_{ab}^n Q_{ab}s_a\cdot s_b} = e^{\frac{1}{2}Nn\beta^2(1-q_1)} e^{\frac{1}{2}\beta^2(q_1-q_0)\sum_{r=0}^{n/m-1}\sum_{a,b=1}^m s_{r+a}\cdot s_{r+b}} e^{\frac{1}{2}\beta^2 q_0\sum_{ab}^n s_a\cdot s_b}$$

2. We now need to make a series of Hubbard–Stratonovich transformations. We already know that

$$e^{\frac{1}{2}\beta^2 q_0\sum_{ab}^n s_a\cdot s_b} = \prod_{i=1}^N \int \frac{dz_i}{\sqrt{2\pi\beta^2 q_0}} e^{-\frac{1}{2}(\beta^2 q_0)^{-1}z_i^2 + z_i\sum_a^n s_{ai}} \tag{4}$$

Show that

$$e^{\frac{1}{2}\beta^2(q_1-q_0)\sum_{r=0}^{n/m-1}\sum_{a,b=1}^m s_{r+a}\cdot s_{r+b}} = \prod_{i=1}^N \prod_{r=0}^{n/m-1} \int \frac{d\zeta_{ri}}{\sqrt{2\pi\beta^2(q_1-q_0)}} e^{-\frac{1}{2}(\beta^2(q_1-q_0))^{-1}\zeta_{ri}^2 + \zeta_{ri}\sum_a^m s_{(r+a)i}}$$

3. After these transformations, the system is diagonalized over the spins. Argue that

$$\sum_{s_1\in\{\pm1\}^N} \cdots \sum_{s_n\in\{\pm1\}^N} e^{\frac{1}{2}\beta^2\sum_{ab}^n Q_{ab}s_a\cdot s_b} = \left[\int \frac{dz}{\sqrt{2\pi\beta^2 q_0}} e^{-\frac{1}{2}(\beta^2 q_0)^{-1}z^2} \times \left(\int \frac{d\zeta}{\sqrt{2\pi\beta^2(q_1-q_0)}} e^{-\frac{1}{2}(\beta^2(q_1-q_0))^{-1}\zeta^2} (2\cosh(z+\zeta) e^{\frac{1}{2}\beta^2(1-q_1)})^m\right)^{n/m}\right]^N$$

4. Now we are in a place where we can take the limit of n to zero. Show that in that limit, the effective action is given by

$$\mathcal{S}(q_0, q_1, m) = \frac{1}{2}\beta^2[1 - (1-m)q_1^2 - mq_0^2] + \frac{1}{m}\int \frac{dz}{\sqrt{2\pi\beta^2 q_0}} e^{-\frac{1}{2}(\beta^2 q_0)^{-1}z^2}$$
$$\times \log\left(\int \frac{d\zeta}{\sqrt{2\pi\beta^2(q_1-q_0)}} e^{-\frac{1}{2}(\beta^2(q_1-q_0))^{-1}\zeta^2}(2\cosh(z+\zeta)e^{\frac{1}{2}\beta^2(1-q_1)})^m\right)$$

5. We can make a few sanity checks. Show that if $m = 0$, $m = 1$, or $q_0 = q_1$, the effective action reverts to that for the replica symmetric approximation.

   Hint: Remember that $p(q_0) = m$ and $p(q_1) = 1 - m$. What integrals can be evaluated in which limits? $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

6. Higher RSB approaches follow a similar pattern to this one. For 2RSB, see if you can convince yourself that

$$\mathcal{S} = \frac{1}{2}\beta^2[1 - (1-m_2)q_2^2 - (m_2 - m_1)q_1^2 - m_1 q_0^2]$$
$$+ \frac{1}{m_1}\int \frac{dz_0}{\sqrt{2\pi\beta^2 q_0}} e^{-\frac{1}{2}(\beta^2 q_0)^{-1}z_0^2} \log\left[\int \frac{dz_1}{\sqrt{2\pi\beta^2(q_1-q_0)}} e^{-\frac{1}{2}(\beta^2(q_1-q_0))^{-1}z_1^2}\right.$$
$$\left.\times\left(\int \frac{dz_2}{\sqrt{2\pi\beta^2(q_2-q_1)}} e^{-\frac{1}{2}(\beta^2(q_2-q_1))^{-1}z_2^2}(2\cosh(z_0+z_1+z_2)e^{\frac{1}{2}\beta^2(1-q_2)})^{m_2}\right)^{m_1/m_2}\right]$$

   Can you guess the form for general kRSB?