7 Nonequilibrium dynamics

DMFT by path integration. By far the most popular way for practitioners to derive DMFT equations in complicated settings is the Martin–Siggia–Rose–De Dominicis–Janssen path integral formalism. Here we will see how this is carried out for our spherical spin glasses with Langevin dynamics

$$\frac{\partial}{\partial t}\mathbf{s}(t) = -\mu(t)\mathbf{s}(t) - \nabla H(\mathbf{s}(t)) + \boldsymbol{\xi}(t) \tag{1}$$

for Gaussian white noise $\boldsymbol{\xi}(t)$ with $\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta(t-t')\delta_{ij}$. The path integral

$$\int_{\mathbf{s}(0)=\mathbf{s}_0} \mathcal{D}\mathbf{s}\, F[\mathbf{s}] \tag{2}$$

integrates a functional F over all trajectories $\mathbf{s}(t)$ with initial condition $\mathbf{s}(0) = \mathbf{s}_0$. From now on we will omit the initial condition notation for brevity, but remember that it is implicitly present.

1. Define the *generating functional*

$$Z[\mathbf{h}] = \int \mathcal{D}\mathbf{s}\, \delta[\boldsymbol{\xi} - \partial_t\mathbf{s} - \mu\mathbf{s} - \nabla H(\mathbf{s}) + \mathbf{h}] \tag{3}$$

where $\mathbf{h}(t)$ is a time-dependent external field we impose, and we have used a functional δ-function which selects only a function which is identically zero. What is the function we have put inside? Argue that $Z[\mathbf{h}] = 1$ always.

2. We now employ the functional Fourier representation of the functional δ-function, with

$$Z[\mathbf{h}] = \int \mathcal{D}\mathbf{s}\,\mathcal{D}\hat{\mathbf{s}}\, e^{\int dt\, \hat{\mathbf{s}}(t)\cdot[\boldsymbol{\xi}(t) - \partial_t\mathbf{s}(t) - \mu(t)\mathbf{s}(t) - \nabla H(\mathbf{s}(t)) + \mathbf{h}(t)]} \tag{4}$$

Using this representation, argue that the response function

$$R(t,t') = \frac{1}{N}\sum_{i=1}^{N}\left\langle \frac{\delta\mathbf{s}(t)}{\delta\mathbf{h}(t')}\right\rangle \tag{5}$$

is given by $R(t,t') = \frac{1}{N}\langle \mathbf{s}(t)\cdot\hat{\mathbf{s}}(t')\rangle$.
Hint: $\mathbf{s}(t) = \int \mathcal{D}\mathbf{s}'\,\delta[\boldsymbol{\xi} - \partial_t\mathbf{s}' - \mu\mathbf{s}' - \nabla H(\mathbf{s}') + \mathbf{h}]\,\mathbf{s}'(t)$

3. In this form our Gaussian averages can be made explicitly. Argue that

$$\langle Z[0]\rangle = \int \mathcal{D}\mathbf{s}\,\mathcal{D}\hat{\mathbf{s}}\, e^{\int dt\, \hat{\mathbf{s}}(t)\cdot[T\hat{\mathbf{s}}(t) - \partial_t\mathbf{s}(t) - \mu(t)\mathbf{s}(t) - \nabla H(\mathbf{s}(t))]} \tag{6}$$

4. Using the Gaussian property of our Hamiltonian, $\overline{H(\mathbf{s})H(\mathbf{s}')} = Nf(\frac{\mathbf{s}\cdot\mathbf{s}'}{N})$, argue that

$$\overline{\langle Z[0] \rangle} = \int \mathcal{D}\mathbf{s}\,\mathcal{D}\hat{\mathbf{s}}\, e^{\int dt\, \hat{\mathbf{s}}(t)\cdot[T\hat{\mathbf{s}}(t) - \partial_t \mathbf{s}(t) - \mu(t)\mathbf{s}(t)]} \tag{7}$$
$$\times\, e^{\frac{N}{2}\int dt\, dt'\, [\frac{\hat{\mathbf{s}}(t)\cdot\hat{\mathbf{s}}(t')}{N} f'(\frac{\mathbf{s}(t)\cdot\mathbf{s}(t')}{N}) + \frac{\mathbf{s}(t)\cdot\hat{\mathbf{s}}(t')}{N}\frac{\mathbf{s}(t')\cdot\hat{\mathbf{s}}(t)}{N} f''(\frac{\mathbf{s}(t)\cdot\mathbf{s}(t')}{N})]}$$

Why don't we need to use replicas?

5. Now we're in a position to define order parameters, with

$$C(t,t') = \frac{1}{N}\mathbf{s}(t)\cdot\mathbf{s}(t') \qquad R(t,t') = \frac{1}{N}\mathbf{s}(t)\cdot\hat{\mathbf{s}}(t') \tag{8}$$
$$D(t,t') = \frac{1}{N}\hat{\mathbf{s}}(t)\cdot\hat{\mathbf{s}}(t') \tag{9}$$

This will be easiest using a 'block matrix' approach. Argue that if

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{s}(t) \\ \hat{\mathbf{s}}(t') \end{bmatrix} \tag{10}$$

then

$$A(t,t') \equiv \frac{1}{N}\mathbf{x}(t)\cdot\mathbf{x}(t')^T = \begin{bmatrix} C(t,t') & R(t,t') \\ R(t',t) & D(t,t') \end{bmatrix} \tag{11}$$

6. Show that

$$\int dt\, \hat{\mathbf{s}}(t)\cdot[T\hat{\mathbf{s}}(t) - \partial_t \mathbf{s}(t) - \mu(t)\mathbf{s}(t)] = -\frac{1}{2}\int dt\, dt'\, \mathbf{x}(t)^T M(t,t')\mathbf{x}(t) \tag{12}$$

for

$$M(t,t') = \begin{bmatrix} 0 & (\mu(t) + \partial_t)\delta(t-t') \\ (\mu(t) - \partial_t)\delta(t-t') & -2T\delta(t-t') \end{bmatrix} \tag{13}$$

Hint: Integration by parts says $\int dt'\, \delta'(t'-t)f(t') = -f'(t)$.

7. By inserting a δ-function defining the order parameter A, making its Fourier representation with conjugate field $\tilde{A}$, and doing a Gaussian integral in $\mathbf{x}$, show that

$$\overline{\langle Z[0] \rangle} \propto \int \mathcal{D}A\,\mathcal{D}\tilde{A}\, e^{\frac{N}{2}\int dt\, dt'\, \operatorname{Tr}\tilde{A}(t,t')A(t,t') - \frac{N}{2}\log\det(\tilde{A}+M) + \frac{N}{2}\mathcal{G}[A]} \tag{14}$$

for

$$\mathcal{G}[A] = \int dt\, dt'\, [D(t,t')f'(C(t,t')) + R(t,t')R(t',t)f''(C(t,t'))] \tag{15}$$

Hint: $\operatorname{Tr} AB = \sum_{ij} A_{ij}B_{ji}$. Don't worry about functional integrals and linear algebra: it all works the same, but 'functional'!

8. Argue that

$$0 = \tilde{A}(t,t') + \frac{\delta \mathcal{G}[A]}{\delta A(t,t')} \qquad 0 = A(t,t') - (\tilde{A} + M)^{-1}(t,t') \tag{16}$$

Combining the two equations, further show that

$$\int dt''\, M(t,t'')A(t'',t') = \int dt''\, \frac{\delta \mathcal{G}[A]}{\delta A(t,t'')} A(t'',t') + I\delta(t,t') \tag{17}$$

Hint: Functional 'multiplication' is convolution, so consider $f^{-1}(t,t')$ as defined by the relation $\delta(t-t') = \int dt''\, f^{-1}(t,t'')f(t'',t')$.

9. We must eventually pay the cost of our compact matrix notation. Inserting the definitions of $\mathcal{G}$, A, and M, show that your four equations (one for each matrix element) are (right to left, top to bottom)

$$(\mu(t) - \partial_t)R(t',t) = \delta(t-t') \tag{18}$$
$$+ \int dt''\, [D(t,t'')f''(C(t,t''))C(t'',t') + R(t'',t)f''(C(t,t''))R(t',t'')]$$
$$(\mu(t) - \partial_t)D(t,t') = \tag{19}$$
$$+ \int dt''\, [D(t,t'')f''(C(t,t''))R(t'',t') + R(t'',t)f''(C(t,t''))D(t'',t')]$$
$$(\mu(t) + \partial_t)C(t,t') = 2TR(t',t) \tag{20}$$
$$+ \int dt''\, [R(t,t'')f''(C(t,t''))C(t'',t') + f'(C(t,t''))R(t',t'')]$$
$$(\mu(t) + \partial_t)R(t,t') = \delta(t-t') \tag{21}$$
$$+ \int dt''\, [R(t,t'')f''(C(t,t''))R(t'',t') + f'(C(t,t''))D(t'',t')]$$

10. We have almost reproduced the equations from the cavity approach, but there is an extra order parameter, $D(t,t')$. First, argue from the equations above that $D(t,t') = 0$ is a solution. Second, show that if we add a time-dependent variance to the noise of the form $\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta(t-t')_{ij} + 2\Gamma(t,t')\delta_{ij}$, then

$$D(t,t') = \frac{1}{N} \frac{\delta\langle Z[0]\rangle}{\delta\Gamma(t,t')}\bigg|_{\Gamma=0} \tag{22}$$

Argue that this implies that D must *always* be zero.

11. Observe that you recover the same equations for C and R as derived via the cavity method. Rejoice.