6 Dynamics: the cavity method

Consider describing our system with Langevin dynamics

$$\dot{\mathbf{s}}(t) = -\mu(t)\mathbf{s}(t) - \frac{\partial H}{\partial \mathbf{s}}(\mathbf{s}) + \boldsymbol{\xi}(t) \tag{1}$$

where $\mu(t)$ is a dynamic Langrange multiplier that keeps the spherical constraint satisfied. We want to understand the dynamical behavior averaged over noise and disorder. Clearly, we cannot just take the average of the equation over disorder: the average gradient of H at any point is zero, so we would loose all influence of the energy! To proceed, we need to use more careful techniques. We will employ a technique called *the cavity method*.

We consider separating the system into two pieces: one spin s_1 and the rest of the system $\mathbf{s}_{\setminus 1}$. The Hamiltonian likewise splits, with

$$H(\mathbf{s}) = H_1(s_1) + H_{\setminus 1}(\mathbf{s}_{\setminus 1}) + H_{\text{int}}(s_1, \mathbf{s}_{\setminus 1}) \tag{2}$$

where

$$H_1(s_1) = \frac{J_{1,\ldots,1} s_1^p}{\sqrt{2p!N^{p-1}}} \tag{3}$$

$$H_{\setminus 1}(\mathbf{s}_{\setminus 1}) = \frac{1}{\sqrt{2p!N^{p-1}}} \sum_{i_1,\ldots,i_p=2}^{N} J_{i_1,\ldots,i_p} s_{i_1} \cdots s_{i_p} \tag{4}$$

$$H_{\text{int}}(s_1, \mathbf{s}_{\setminus 1}) = \frac{p}{\sqrt{2p!N^{p-1}}} \sum_{i_1,\ldots,i_{p-1}=1}^{N} J_{1,i_1,\ldots,i_{p-1}} s_1 s_{i_1} \cdots s_{i_{p-1}} \tag{5}$$

Note that $H_{\setminus 1}$ is just the Hamiltonian of a spherical model with $N-1$ spins, and that H_{int} depends only linearly on s_1 to highest order in N. The equation for the one spin is

$$\begin{aligned}\dot{s}_1(t) &= -\mu(t)s_1(t) - \frac{\partial H_1}{\partial s_1}(s_1) - \frac{\partial H_{\text{int}}}{\partial s_1}(s_1, \mathbf{s}_{\setminus 1}) + \xi_1(t) \\ &\simeq -\mu(t)s_1(t) - \frac{\partial H_{\text{int}}}{\partial s_1}(\mathbf{s}_{\setminus 1}) + \xi_1(t)\end{aligned} \tag{6}$$

where we have used the linearity of H_{int} to write its gradient as a function of $\mathbf{s}_{\setminus 1}$ alone, and the fact that H_1 is smaller than H_{int} by factors of N. Likewise, the equation for the rest of the spins is

$$\begin{aligned}\dot{\mathbf{s}}_{\setminus 1}(t) &= -\mu(t)\mathbf{s}_{\setminus 1}(t) - \frac{\partial H_{\setminus 1}}{\partial \mathbf{s}_{\setminus 1}}(\mathbf{s}_{\setminus 1}) - \frac{\partial H_{\text{int}}}{\partial \mathbf{s}_{\setminus 1}}(s_1, \mathbf{s}_{\setminus 1}) + \boldsymbol{\xi}_{\setminus 1}(t) \\ &\simeq -\mu(t)\mathbf{s}_{\setminus 1}(t) - \frac{\partial H_{\setminus 1}}{\partial \mathbf{s}_{\setminus 1}}(\mathbf{s}_{\setminus 1}) - s_1 \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial \mathbf{s}_{\setminus 1}}(\mathbf{s}_{\setminus 1}) + \boldsymbol{\xi}_{\setminus 1}(t)\end{aligned}$$

where in the second line we have used the approximate linearity of H_{int} to write it as s_1 times the derivative with respect to s_1.

We now make a key assumption: because there are very many spins, the effect of s_1 on the others can be treated as a small perturbation. We see that the effect of s_1 is already in the form a time-dependent perturbation on the equation of motion of $\dot{\mathbf{s}}_{\setminus 1}$, of the form

$$\mathbf{h}(t) = -s_1 \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial \mathbf{s}_{\setminus 1}}(\mathbf{s}_{\setminus 1}(t)) \tag{7}$$

It follows that the variation of the other spins caused by s_1 is

$$\begin{aligned}
\frac{\delta \mathbf{s}_{\setminus 1}(t)}{\delta s_1(t')} &= \int dt'' \sum_{i=1}^{N} \frac{\delta \mathbf{s}_{\setminus 1}(t)}{\delta h_i(t'')} \frac{\delta h_i(t'')}{\delta s_1(t')} \\
&= -\int dt'' \sum_{i=2}^{N} \frac{\delta \mathbf{s}_{\setminus 1}(t)}{\delta h_i(t'')} \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial s_i}(\mathbf{s}_{\setminus 1}(t''))\delta(t''-t') \\
&= -\sum_{i=2}^{N} \frac{\delta \mathbf{s}_{\setminus 1}(t)}{\delta h_i(t')} \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial s_i}(\mathbf{s}_{\setminus 1}(t'))
\end{aligned}$$

We can see the response function

$$R(t,t') = \frac{1}{N} \sum_{i=1}^{N} \left\langle \frac{\delta s_i(t)}{\delta h_i(t')} \right\rangle \tag{8}$$

trying to emerge! Let $\mathbf{s}^0_{\setminus 1}(t)$ be the trajectory of the rest of the spins in the complete absence of s_1. Then linear response tells us that the effect of its presence will be

$$\delta \mathbf{s}_{\setminus 1}(t) = \int dt' \frac{\delta \mathbf{s}_{\setminus 1}(t)}{\delta s_1(t')} s_1(t') = -\int dt' \sum_{i=2}^{N} \frac{\delta \mathbf{s}_{\setminus 1}(t)}{\delta h_i(t')} \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial s_i}(\mathbf{s}_{\setminus 1}(t')) s_1(t') \tag{9}$$

Now we use this to expand the dependence of the equation of motion of s_1 on $\mathbf{s}_{\setminus 1}$:

$$\begin{aligned}
\dot{s}_1(t) &= -\mu(t)s_1(t) - \frac{\partial H_{\text{int}}}{\partial s_1}(\mathbf{s}_{\setminus 1}(t)) + \xi_1(t) \\
&= -\mu(t)s_1(t) - \frac{\partial H_{\text{int}}}{\partial s_1}(\mathbf{s}^0_{\setminus 1}(t) + \delta \mathbf{s}_{\setminus 1}(t)) + \xi_1(t) \\
&= -\mu(t)s_1(t) - \frac{\partial H_{\text{int}}}{\partial s_1}(\mathbf{s}^0_{\setminus 1}(t)) - \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial \mathbf{s}_{\setminus 1}}(\mathbf{s}_{\setminus 1}(t)) \cdot \delta \mathbf{s}_{\setminus 1}(t) + \xi_1(t) \\
&= -\mu(t)s_1(t) - \frac{\partial H_{\text{int}}}{\partial s_1}(\mathbf{s}^0_{\setminus 1}(t)) + \int dt' \sum_{i,j=2}^{N} \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial s_i}(\mathbf{s}_{\setminus 1}(t)) \frac{\delta s_i(t)}{\delta h_j(t')} \frac{\partial^2 H_{\text{int}}}{\partial s_1 \partial s_j}(\mathbf{s}_{\setminus 1}(t')) s_1(t') + \xi_1(t)
\end{aligned}$$

Now we are in a place where we have make averages over disorder. There are two nontrivial contributions: the first term has mean zero but contributes to the noise on the spin on the same level as ξ. To characterize it, we need covariances with respect to H_{int}, which are

$$\begin{aligned}\overline{H_{int}(\mathbf{s})H_{int}(\mathbf{s}')} &= \frac{p^2}{2p!N^{p-1}} \sum_{i_2,\dots,i_p} \sum_{j_2,\dots,j_p} \overline{J_{1,i_2,\dots,i_p} J_{1,j_2,\dots,j_p}} s_1 s_{i_2} \cdots s_{i_p} s'_1 s'_{j_2} \cdots s'_{j_p} \\ &= \frac{1}{2} p s_1 s'_1 \left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right)^{p-1} = s_1 s'_1 f'\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right)\end{aligned}$$

The term $\frac{\partial H_{int}}{\partial s_1}(\mathbf{s}^0_{\setminus 1}(t))$ is therefore Gaussian with zero mean and covariance

$$\overline{\frac{\partial H_{int}}{\partial s_1}(\mathbf{s})\frac{\partial H_{int}}{\partial s_1}(\mathbf{s}')} = \frac{\partial}{\partial s_1}\frac{\partial}{\partial s'_1}\overline{H_{int}(\mathbf{s})H_{int}(\mathbf{s}')} = f'\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right) \tag{10}$$

The second term depends on

$$\frac{\partial^2 H_{int}}{\partial s_1 \partial s_i}(\mathbf{s})\frac{\partial^2 H_{int}}{\partial s_1 \partial s_j}(\mathbf{s}') \tag{11}$$

which is averaged to give

$$\begin{aligned}\overline{\frac{\partial^2 H_{int}}{\partial s_1 \partial s_i}(\mathbf{s})\frac{\partial^2 H_{int}}{\partial s'_1 \partial s'_j}(\mathbf{s}')} &= \frac{\partial^4}{\partial s_1 \partial s'_1 \partial s_i \partial s'_j}\overline{H_{int}(\mathbf{s})H_{int}(\mathbf{s}')} \\ &= \frac{\partial^2}{\partial s_i \partial s'_j} f'\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right) = \frac{\partial}{\partial s_i}\frac{s_j}{N} f''\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right) \\ &= \frac{1}{N}\delta_{ij} f''\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right) + \frac{1}{N^2} s'_i s_j f'''\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right) \simeq \frac{1}{N}\delta_{ij} f''\left(\frac{\mathbf{s}\cdot\mathbf{s}'}{N}\right)\end{aligned} \tag{12}$$

where we have neglected a term smaller in N. We can therefore write

$$\dot{s}_1(t) = -\mu(t)s_1(t) + \int dt' \left(\frac{1}{N}\sum_{i=2}^{N}\frac{\delta s_i(t)}{\delta h_j(t')}\right) f''\left(\frac{\mathbf{s}(t)\cdot\mathbf{s}(t')}{N}\right) s_1(t') + \Xi_1(t)$$

where Ξ is a Gaussian noise with mean $\overline{\Xi_1(t)} = 0$ and variance

$$\overline{\Xi(t)\Xi(t')} = 2T\delta_{ij}\delta(t-t') + f'\left(\frac{\mathbf{s}(t)\cdot\mathbf{s}(t')}{N}\right) \tag{13}$$

We now recognize that this disordered-averaged equation for the cavity spin depends only on the rest of the system through its correlation and response functions. Because at large N behavior should not change by adding or removing one spin, we assume that the correlation and response functions for the $N-1$ spin system are the same as those for the whole system. We therefore have

$$\dot{s}_1(t) = -\mu(t)s_1(t) + \int dt'\, R(t,t') f''\left(C(t,t')\right) s_1(t') + \Xi_1(t)$$

where Ξ is a Gaussian noise with mean $\overline{\Xi_1(t)} = 0$ and variance

$$\overline{\Xi(t)\Xi(t')} = 2T\delta_{ij}\delta(t-t') + f'\left(C(t,t')\right) \tag{14}$$

The result is a one-dimensional stochastic equation that removes the dependence of the rest of the system at the cost of *memory*: the equation is no longer local in time, but depends in both the noise and in the force on the configuration at previous times! This is a self-consistent way of treating the fact that the system and cavity influence each other and have feedback, one on the other.

To proceed, we note the following: we could have make this argument for any one of the spins, not just s_1. Therefore, the same equation is true for all of them. We can therefore write

$$\dot{\mathbf{s}}(t) = -\mu(t)\mathbf{s}(t) + \int dt'\, R(t,t')f''\left(C(t,t')\right)\mathbf{s}(t') + \boldsymbol{\Xi}(t)$$

We can therefore build the correlation and response functions the same way we did in the non-disordered case. For the correlation function, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}C(t,t') = \frac{1}{N}\langle \dot{\mathbf{s}}(t)\cdot\mathbf{s}(t')\rangle && (15)\\
&= -\mu(t)\frac{1}{N}\langle \mathbf{s}(t)\cdot\mathbf{s}(t')\rangle + \int dt''\, R(t,t'')f''\left(C(t,t'')\right)\frac{1}{N}\langle \mathbf{s}(t'')\cdot\mathbf{s}(t')\rangle + \frac{1}{N}\langle \boldsymbol{\Xi}(t)\cdot\mathbf{s}(t')\rangle\\
&= -\mu(t)C(t,t') + \int dt''\, R(t,t'')f''\left(C(t,t'')\right)C(t',t'') + 2TR(t',t) + \int dt''\, f'(C(t,t''))R(t',t'')
\end{aligned}$$

where in the last line we have used Wick's theorem to see that

$$\begin{aligned}
\frac{1}{N}\langle \boldsymbol{\Xi}(t)\cdot\mathbf{s}(t')\rangle &= \frac{1}{N}\int dt'' \sum_{ij}\langle \Xi_i(t)\Xi_j(t'')\rangle \left\langle \frac{\delta s_i(t')}{\delta \Xi_j(t'')}\right\rangle && (16)\\
&= \int dt''\left(2T\delta(t-t'') + f'\left(C(t,t'')\right)\right)R(t',t'')\\
&= 2TR(t',t) + \int dt''\, f'(C(t,t''))R(t',t'')
\end{aligned}$$

Likewise, the equation for the response function is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}R(t,t') = \frac{1}{N}\sum_{i=1}^{N}\left\langle \frac{\delta \dot{s}_i(t)}{\delta \Xi_i(t')}\right\rangle && (17)\\
&= -\mu(t)R(t,t') + \int dt''\, R(t,t'')f''\left(C(t,t'')\right)R(t'',t') + \delta(t-t')
\end{aligned}$$

where we have used $\frac{\delta \Xi_i(t)}{\delta \Xi_i(t')} = \delta(t-t')$.

We have used a self-consistent argument to write the equations for the correlation and response function averaged over the noise and disorder. These equations are *integro-differential* equations, which involve memory integrals over all times. To treat the equilibrium dynamics, we assert that the behavior should be time-translation invariant. The equations are then

$$
\begin{aligned}
C'(\tau) &= -\mu C(\tau) + 2TR(-\tau) + \int d\tau' \, R(\tau-\tau') f''\left(C(\tau-\tau')\right) C(\tau') + \int d\tau' \, f'(C(\tau-\tau'))R(-\tau') \\
R'(\tau) &= -\mu R(\tau) + \int dt'' \, R(\tau-\tau') f''\left(C(\tau-\tau')\right) R(\tau') + \delta(\tau)
\end{aligned}
$$

Keeping in mind that the response function will only be nonzero for positive time differences, we can put limits on the integrals, and have

$$
\begin{aligned}
C'(\tau) &= -\mu C(\tau) + 2TR(-\tau) + \int_{-\infty}^{\tau} d\tau' \, R(\tau-\tau') f''\left(C(\tau-\tau')\right) C(\tau') + \int_{-\infty}^{0} d\tau' \, f'(C(\tau-\tau'))R(-\tau') \\
R'(\tau) &= -\mu R(\tau) + \int_{0}^{\tau} dt'' \, R(\tau-\tau') f''\left(C(\tau-\tau')\right) R(\tau') + \delta(\tau)
\end{aligned}
$$

We might come up with a scheme to solve the second equation, where we start from $\tau = 0$ and make little steps forward, each time calculating the derivative at the current time difference by integrating over what has already passed. But the first equation appears to depend on memory of *all previous times to infinity*. This could be difficult! However, in fact the negative time difference dependence exactly cancels between the two terms in that equation. This can be seen because equilibrium dynamics follows the fluctuation dissipation relation $R(\tau) = -\beta\Theta(\tau)C'(\tau)$. Noting that

$$
\beta \frac{\partial}{\partial \tau'} f'(C(\tau-\tau')) = -\beta f''(C(\tau-\tau'))C'(\tau-\tau') = f''(C(\tau-\tau'))R(\tau-\tau') \quad (18)
$$

we can integrate the first expression by parts to find

$$
\begin{aligned}
C'(\tau) &= -\mu C(\tau) + 2TR(-\tau) - \beta \int_{-\infty}^{\tau} d\tau' \, f'\left(C(\tau-\tau')\right) C'(\tau') + \int_{-\infty}^{0} d\tau' \, f'(C(\tau-\tau'))R(-\tau') \\
&= -\mu C(\tau) + 2TR(-\tau) - \beta \int_{-\infty}^{\tau} d\tau' \, f'\left(C(\tau-\tau')\right) C'(\tau') - \beta \int_{-\infty}^{0} d\tau' \, f'(C(\tau-\tau'))C'(-\tau') \\
&= -\mu C(\tau) + 2TR(-\tau) - \beta \int_{-\infty}^{\tau} d\tau' \, f'\left(C(\tau-\tau')\right) C'(\tau') + \beta \int_{-\infty}^{0} d\tau' \, f'(C(\tau-\tau'))C'(\tau') \\
&= -\mu C(\tau) + 2TR(-\tau) - \beta \int_{0}^{\tau} d\tau' \, f'\left(C(\tau-\tau')\right) C'(\tau')
\end{aligned}
$$

where we have also used the fact that $C(\tau)$ is symmetric. This is almost a closed equation in C alone! We can eliminate R by noting that it functions only to set the derivative of C at $\tau = 0$. We know that $R(0_+) = 1$ and $R(0_-) = 0$. We must have $C(\tau)$ symmetric, so that

$$
C'(0_+) = -\mu \qquad C'(0_-) = -\mu + 2T \qquad C'(0_+) = -C'(0_-) \quad (19)
$$

It follows that $\mu = T$ and that $C'(0_+) = T$. We therefore have for $\tau > 0$,

$$C'(\tau) = -TC(\tau) - \beta \int_0^\tau d\tau' \, f'\left(C(\tau-\tau')\right) C'(\tau')$$

This equation can be solved iteratively in the way noted above. The result is a transition between ergodic behavior for large temperatures and nonergodic behavior for small ones, with a distinctive transition in between. However, we can analytically determine where a phase transition takes place. We do this through the Laplace transform,

$$\mathcal{L}[g](s) = \tilde{g}(s) = \int_0^\infty d\tau \, e^{-\tau s} g(\tau) \tag{20}$$

To apply this, we need to make use of a few properties of the Laplace transform:

$$\mathcal{L}[g'](s) = s\tilde{g}(s) - g(0) \qquad\qquad \mathcal{L}[g * h](s) = \tilde{g}(s)\tilde{h}(s) \tag{21}$$

We therefore have

$$s\tilde{C}(s) - 1 = -T\tilde{C}(s) - \beta\mathcal{L}[f' \circ C](s)(s\tilde{C}(s) - 1) \tag{22}$$

where we have used notation of convolution for $(f' \circ C)(\tau) = f'(C(\tau))$. Rearranging, we have

$$\beta^2\mathcal{L}[f' \circ C](s) = \beta - \frac{\tilde{C}(s)}{s\tilde{C}(s) - 1} = \beta + \frac{1}{s}\left(\frac{1}{1 - s\tilde{C}(s)} - 1\right) \tag{23}$$

Now suppose that the correlation approaches some constant q at long times, with $C(\tau) = q + \epsilon(\tau)$. This would result in $\tilde{C}(s) = \frac{q}{s} + \tilde{\epsilon}(s)$. We can expand the Laplace transform of the convolution as well, with

$$\begin{aligned}
\mathcal{L}[f' \circ C](s) &= \int_0^\infty d\tau \, e^{-\tau s} f'(C(\tau)) &(24)\\
&= \int_0^\infty d\tau \, e^{-\tau s} \left[f'(q) + f''(q)\epsilon(\tau) + \frac{1}{2}f'''(q)\epsilon(\tau)^2 + \cdots\right] \\
&= \frac{f'(q)}{s} + f''(q)\tilde{\epsilon}(s) + \frac{1}{2}f'''(q)\mathcal{L}[\epsilon^2](s) + \cdots &(25)
\end{aligned}$$

Large times in the time domain correspond to small s in the Laplace domain. We can therefore analyze asymptotically long times by expanding about small s. Doing this to the equation above, we find

$$\beta^2 f'(q)\frac{1}{s} + \beta^2 f''(q)\tilde{\epsilon}(0) + \frac{1}{2}f'''(q)\mathcal{L}[\epsilon^2](0) + \cdots + O(s) = \frac{q}{1-q}\frac{1}{s} + \beta + \frac{\tilde{\epsilon}(0)}{(1-q)^2} + O(s)$$

We see by matching the s^{-1} term that $\beta^2 f'(q) = \frac{q}{1-q}$, which is the condition for q_1 in the 1RSB state. For s beyond this power, the left-hand side in principle has many terms, but near any transition things simplify. At a transition, decay timescales diverge, and $\tilde{\epsilon}(0)$ likewise diverges. Moreover, $\tilde{\epsilon}(0)$ diverges faster than $\mathcal{L}[\epsilon^n](0)$ for $n > 1$ because higher powers decay faster at large time. We therefore can gather the terms with s^0 proportional to $\tilde{\epsilon}(0)$, which gives $\beta^2 f''(q) = (1-q)^{-2}$. Recalling the last lecture, this second condition is precisely the edge of stability for the 1RSB state. Therefore, the value of q and T corresponding to a dynamic phase transition are precisely the q_d and T_d we calculated last time for the overlap and temperature at which the 1RSB phase becomes stable. This is a temperature *greater* than the 1RSB transition temperature, and therefore a dynamic transition occurs and ergodicity is broken without a thermodynamic phase transition occurring!

When the equations are numerically integrated, one finds that for temperatures well higher than the dynamic transition temperature the correlation function quickly decays to zero. As the dynamic transition is approached, a step begins to develop at q_d where the correlation function stalls, and this the length of time at which the correlation function stays at this step diverges as the transition is reached. Above the transition, the step continues to move along with the value of q stable in the corresponding 1RSB state.

How is it that we have an ergodicity breaking phase transition in the absence of a thermodynamic transition? We have shown that there is a range of temperatures where the Boltzmann measure describes a single cluster of states, but long-time dynamics cannot explore this cluster. We will see next time that some of this tension is relieved when we look at the *out-of-the-equilibrium* dynamics rather than the equilibrium ones, and we will see a mechanism for what happens in this strange regime.