7 Dynamics: out of equilibrium

In the previous lecture we derived from the Langevin equation

$$\dot{\mathbf{s}}(t) = -\mu(t)\mathbf{s}(t) - \frac{\partial H}{\partial \mathbf{s}}(\mathbf{s}(t)) + \boldsymbol{\xi}(t) \tag{1}$$

the equations governing the correlation and response functions:

$$\begin{aligned}
&\frac{\partial}{\partial t}C(t,t') = -\mu(t)C(t,t') + 2TR(t',t) \qquad (2)\\
&+\int_{-\infty}^{t} dt''\, R(t,t'')f''(C(t,t''))C(t'',t') + \int_{-\infty}^{t'} dt''\, f'(C(t,t''))R(t',t'')\\
&\frac{\partial}{\partial t}R(t,t') = -\mu(t)R(t,t') + \int_{t'}^{t} dt''\, R(t,t'')f''(C(t,t''))R(t'',t') + \delta(t-t')
\end{aligned}$$

Imagine the following scenario: we have the system at equilibrium at some temperature T' for all time under $t=0$, at which point we quench the system by lowering the temperature to T. For all times t and t' less than zero, the system is in equilibrium and the fluctuation theorem holds, and the cancellation for the negative-time part of the integrals in the equation for C can be made again, but this time with the boundary term contributing, which gives

$$\begin{aligned}
&\frac{\partial}{\partial t}C(t,t') = -\mu(t)C(t,t') + 2TR(t',t) + \beta' f'(C(0,t))C(0,t') \qquad (3)\\
&+\int_{0}^{t} dt''\, R(t,t'')f''(C(t,t''))C(t',t'') + \int_{0}^{t'} dt''\, f'(C(t,t''))R(t',t'')
\end{aligned}$$

In this form, the equations can be solved numerically on a two-time grid. It's extremely rare that two-time equations like these can be explicitly solved. However, today we will see that there *is* a solution to these equations in the limit of both t and t' to long times. This is the aging solution.

Before we get to solving the equations, we supplement the equations above with one for the Langrange multiplier $\mu(t)$. For pure models, $\mu(t)$ can be directly related to the expectation value of the instantaneous energy density. First, we write as before

$$\begin{aligned}
\frac{\partial}{\partial t}C(t,t') &= \frac{1}{N}\langle \dot{\mathbf{s}}(t)\cdot\mathbf{s}(t')\rangle \qquad (4)\\
&= -\mu(t)C(t,t') - \frac{1}{N}\left\langle \frac{\partial H}{\partial \mathbf{s}}(\mathbf{s}(t))\cdot\mathbf{s}(t')\right\rangle + 2TR(t',t)
\end{aligned}$$

Because the Hamiltonian is a homogeneous polynomial, $\mathbf{s}\cdot\frac{\partial H}{\partial \mathbf{s}}(\mathbf{s}) = pH(\mathbf{s})$. Therefore, if we take the limit of $t' \to t$ from above and below, we have the equations

$$\frac{\partial}{\partial t} C(t, t+\epsilon) = -\mu(t) - pE(t) + 2T \tag{5}$$

$$\frac{\partial}{\partial t} C(t, t-\epsilon) = -\mu(t) - pE(t) \tag{6}$$

where $E(t) = \frac{1}{N}\langle H(t)\rangle$ is the average energy density. Summing the two equations, using the fact that $C(t, t')$ is symmetric about $t = t'$, we have

$$\mu(t) = T - pE(t) \tag{7}$$

so that the Lagrange multiplier is directly related to the energy density. We can make another equation for $\mu(t)$ by evaluating the equation for the correlation function as $t' \to t$ from below, which gives

$$\begin{aligned} -T &= -\mu(t) + \int_0^t dt'' \, R(t, t'') f''(C(t, t''))C(t'', t) + \int_0^t dt'' \, f'(C(t, t''))R(t, t'') \\ &= -\mu(t) + \int_0^t dt'' \, R(t, t'')[f''(C(t, t''))C(t, t'') + f'(C(t, t''))] \end{aligned} \tag{8}$$

and therefore

$$\mu(t) = T + \int_0^t dt'' \, R(t, t'')[f''(C(t, t''))C(t, t'') + f'(C(t, t''))] \tag{9}$$

The core insight of the aging solution is to generalize the FDT. Normally, FDT relates the response to the derivative of the correlation, with

$$R(t, t') = \beta \frac{\partial}{\partial t'} C(t, t') \tag{10}$$

or to the integrated response

$$\chi(t, t') = \int_{t'}^t dt'' \, R(t, t'') \tag{11}$$

by

$$\chi(t, t') = \beta \int_{t'}^t dt'' \, \frac{\partial}{\partial t''} C(t, t'') = \beta[C(t, t) - C(t, t')] = \beta[1 - C(t, t')] \tag{12}$$

In some sense, this is a deep statement about the possible ways the system can behave: no matter how the dynamics goes or what perturbation is used, if we measure $C(t, t')$ and $\chi(t, t')$ and make a parametric plot of χ versus C, the result will always lie on a line of slope $-\beta$. In a real sense, we can understand the expected response of the system knowing just the value of the correlation: the timescale at which either happens is irrelevant.

The aging ansatz makes the following guess: that we can find solutions to the equations above that obey a *generalized* FDT, where

$$R(t,t') = \beta X(C(t,t')) \frac{\partial}{\partial t'} C(t,t') \tag{13}$$

for some function X of the value of the correlation alone. This still implies the powerful relation discussed before: if we parametrically plot integrated response and correlation together, we will still have them fall on the same curve, but now potentially not a line. That curve will be given by

$$\chi(t,t') = \beta \int_{t'}^{t} dt'' X(C(t,t'')) \frac{\partial}{\partial t''} C(t,t'') = \beta \int_{C(t,t')}^{1} dC'' X(C'') \tag{14}$$

where we used $dC = \frac{\partial C}{\partial t''} dt''$. Therefore χ is still a simple function of C, with $\chi = \beta F(C)$ for F defined by

$$F(C) = \int_{C}^{1} dC' X(C') \tag{15}$$

For the spherical p-spin models, we make the following ansatz: that X takes two values,

$$X(C) = \begin{cases} 1 & C > q \\ x & C < q \end{cases} \tag{16}$$

The idea is the following: on short timescales when correlations between configurations are high, FDT is approximately satisfied and dynamics are approximately stationary. However, on longer timescales the dynamics is not stationary and FDT is modified to have a different slope. Inserting this ansatz, we will have to find the new slope x and the place where the slope changes q by solving the equations in the appropriate limit.

What is the appropriate limit? We expect this solution to be accurate for $t \to \infty$, after a very long waiting time. Then, we expect the stationary-like solution when $t - t' \ll t$, and the new 'aging' solution when $0 < t'/t < 1$, on the order one. Since our solution is only found in the limit of very large t, the Lagrange multiplier $\mu(t)$ will have approached an asymptotic constant. We can work out this constant directly from this ansatz, with no further assumptions on C and R. We have

$$\begin{aligned}\mu_\infty &= T + \int_0^t dt'\, R(t,t')[f''(C(t,t'))C(t,t') + f'(C(t,t'))] \qquad (17)\\ &= T + \beta \int_0^t dt'\, [f''(C(t,t'))C(t,t') + f'(C(t,t'))]X(C(t,t'))\frac{\partial}{\partial t'}C(t,t')\\ &= T + \beta \int_0^1 dC\, [f''(C)C + f'(C)]X(C)\\ &= T + \beta \int_0^1 dC\, X(C)\frac{\partial}{\partial C}[f'(C)C]\\ &= T + \beta \left[x\int_0^q dC\, \frac{\partial}{\partial C}[f'(C)C] + \int_q^1 dC\, \frac{\partial}{\partial C}[f'(C)C]\right]\\ &= T + \beta[xf'(q)q + (f'(1) - f'(q)q)] = T + \beta f'(1) - \beta(1-x)f'(q)q\end{aligned}$$

where we have again used $dC = \frac{\partial C}{\partial t'}\, dt'$.

To treat the two-time equation for C in the same way, we will need one more feature of the aging solution. This comes from noting the following: since $C(t,t')$ is a monotonic function of t', we can invert it, and write $t' = g(C(t,t'),t)$ for some function g. We can therefore also write the correlation at two times $C(t,t'')$ as a function of the correlations $C(t,t')$ and $C(t'',t')$ with an intermediate time t, using

$$C(t,t'') = C(t, g(C(t',t''),t')) = C(t, g(C(t',t''), g(C(t,t'),t))) \qquad (18)$$

which is a function only of $C(t',t'')$, $C(t,t')$, and t. Since we are interested in asymptotic behavior after long waiting times, we take the limit of this compound function in the limit of t to infinity, which can be checked is well-defined. If such a limit is taken we can write

$$C(t,t'') = c(C(t,t'), C(t',t'')) \qquad (19)$$

for some function c. By manipulation of expressions like those above, this function can be shown to have many properties: it is associative, with $c(c(x,y),z) = c(x,c(y,z))$; it satisfies $x = c(x,1) = c(1,x)$ and $0 = c(x,0) = c(0,y)$; and $c(x,y) \leqslant \min(x,y)$.[1] In fact, for our p-spin models it can be shown by studying the final solution that $c(x,y) = \min(x,y)$ if x and y lie in different time sectors, e.g., if $x < q$ and $y > q$. Note that in general, the properties above can be proven to either result in min or in something isomorphic to multiplication, i.e.,

$$c(x,y) = \mathcal{J}^{-1}(\mathcal{J}(x)\mathcal{J}(y)) \qquad (20)$$

for some function $\mathcal{J}$.

[1] $y = c(1,y) \geqslant c(x,y)$, $x = c(x,1) \geqslant c(x,y)$, QED

Now we turn our attention to the equation for the correlation function, which for $t' < t$ and with $t \to \infty$ is

$$\frac{\partial}{\partial t} C(t,t') = -\mu_\infty C(t,t') + \int_0^t dt''\, R(t,t'') f''(C(t,t''))C(t',t'') + \int_0^{t'} dt''\, f'(C(t,t''))R(t',t'')$$

The integrals from 0 to t' can be written

$$\begin{aligned} &\int_0^{t'} dt''\, [R(t,t'')f''(C(t,t''))C(t',t'') + f'(C(t,t''))R(t',t'')] \qquad (21)\\ &= \beta \int_0^{t'} dt'' \left[f''(C(t,t''))C(t',t'')X(C(t,t''))\frac{\partial}{\partial t''}C(t,t'') + f'(C(t,t''))X(C(t',t''))\frac{\partial}{\partial t''}C(t',t'') \right] \end{aligned}$$

We want to write the integral over $C'' = C(t,t'')$, and $C = C(t,t')$ which is a constant. However, it also depends on $C' = (t',t'')$. Using our fancy asymptotic mapping between correlations, we can write this as $c(C, C'')$ and have

$$\beta \int_0^C dC'' \left[f''(C'')c(C,C'')X(C'') + f'(C'')X(c(C,C''))\frac{\partial c(C,C'')}{\partial C''} \right] \qquad (22)$$

where we have used the fact that $C(t',t'') = c(C(t,t'), C(t',t'')) = c(C,C'')$, and that at the limit of integrations, we have $C(t,0) = 0$ and $C(t,t') = C$. Following the same reasoning, the part of the first integral between t' and t can be written

$$\int_{t'}^t dt''\, R(t,t'')f''(C(t,t''))C(t',t'') = \beta \int_C^1 dC''\, f''(C'')X(C'')c(C,C'') \qquad (23)$$

We can evaluate these integrals in each of the two dynamic sectors. First, when $t - t' \ll t$, the stationary sector, we have $C \geqslant q$. We split the first integral into two pieces corresponding to the different behaviors of $X(C)$:

$$\begin{aligned} &\beta \int_0^C dC'' \left[f''(C'')X(C'')c(C,C'') + f'(C'')X(c(C,C''))\frac{\partial c(C,C'')}{\partial C''} \right] \qquad (24)\\ &= \beta x \int_0^q dC''\, [f''(C'')C'' + f'(C'')] + \beta \int_q^C dC'' \left[f''(C'')c(C,C'') + f'(C'')\frac{\partial c(C,C'')}{\partial C''} \right]\\ &= \beta x \int_0^q dC''\, \frac{\partial}{\partial C''}[f'(C'')C''] + \beta \int_q^C dC''\, \frac{\partial}{\partial C''}[f'(C'')c(C,C'')]\\ &= \beta x f'(q)q + \beta(f'(C)c(C,C) - f'(q)c(C,q))\\ &= \beta x f'(q)q + \beta(f'(C)c(C,C) - f'(q)q) \end{aligned}$$

The second integral has only one piece, since $C > q$, and we have

$$\beta \int_C^1 dC''\, f''(C'') X(C'') c(C, C'') = \beta \int_C^1 dC''\, f''(C'') c(C, C'') \tag{25}$$
$$= \beta \int_C^1 dC'' \frac{\partial}{\partial C''}[f'(C'') c(C, C'')] - \beta \int_C^1 dC''\, f'(C'') \frac{\partial c(C, C'')}{\partial C''}$$
$$= \beta[f'(1) c(C, 1) - f'(C) c(C, C)] - \beta \int_C^1 dC''\, f'(C'') \frac{\partial c(C, C'')}{\partial C''}$$
$$= \beta[f'(1) C - f'(C) c(C, C)] - \beta \int_C^1 dC''\, f'(C'') \frac{\partial c(C, C'')}{\partial C''}$$

We therefore have in this regime

$$\frac{\partial}{\partial t} C(t, t') = -(\mu_\infty - \beta f'(1)) C(t, t') - \beta(1 - x) f'(q) q - \beta \int_{C(t,t')}^1 dC''\, f'(C'') \frac{\partial c(C(t, t'), C'')}{\partial C''}$$

We can write the last term

$$-\beta \int_{C(t,t')}^1 dC''\, f'(C'') \frac{\partial c(C(t, t'), C'')}{\partial C''} \tag{26}$$
$$= -\beta \int_{t'}^t dt'' \frac{dC(t, t'')}{dt''} f'(C(t, t'')) \frac{\partial t''}{\partial C(t, t'')} \frac{\partial}{\partial t''} C(t', t'')$$
$$= -\beta \int_{t'}^t dt''\, f'(C(t, t'')) \frac{\partial}{\partial t''} C(t', t'')$$

which is exactly the same form we had in the time translation invariant problem. Since in this regime we expect $C(t, t') = C_{st}(t - t')$, we can write

$$\frac{\partial}{\partial t} C_{st}(\tau) = -(\mu_\infty - \beta f'(1)) C_{st}(\tau) - \beta(1 - x) f'(q) q - \beta \int_0^\tau d\tau'\, f'(C_{st}(\tau - \tau')) C'_{st}(\tau')$$
$$= -T C_{st}(\tau) - \beta(1 - x) f'(q) q (1 - C_{st}(\tau)) - \beta \int_0^\tau d\tau'\, f'(C_{st}(\tau - \tau')) C'_{st}(\tau')$$

where we inserted the value of μ_∞ we calculated previously. We can solve for the value of the plateau implied by this equation the same way we did for equilibrium. Repeating that argument gives

$$\beta^2 f'(q)[1 - (1 - x) q] = \frac{q}{1 - q} \tag{27}$$

which reproduces the condition of equilibrium when $x = 1$.

When $0 < t/t' < 1$, we are in the aging sector, and we have $0 \leqslant C \leqslant q$. The first integral is now entirely in the regime where $C'' \leqslant q$, and we have

$$\beta \int_0^C dC'' \left[f''(C'')X(C'')c(C,C'') + f'(C'')X(c(C,C'')) \frac{\partial c(C,C'')}{\partial C''} \right] \tag{28}$$
$$= \beta x \int_0^C dC'' \left[f''(C'')c(C,C'') + f'(C'') \frac{\partial c(C,C'')}{\partial C''} \right]$$
$$= \beta x \int_0^C dC'' \frac{\partial}{\partial C''} [f'(C'')c(C,C'')]$$
$$= \beta x f'(C)c(C,C)$$

However, we have to be careful with derivatives of $c(C, C'')$ in the tiny region where $C'' \simeq C$, since this corresponds in the integral to $C(t', t'')$ where t'' goes to t' and therefore goes to one. Excising this part to treat separately, we have

$$\beta \int_{C-\epsilon}^C dC'' \left[f'(C'')X(c(C,C'')) \frac{\partial c(C,C'')}{\partial C''} \right] \tag{29}$$
$$= \beta f'(C) \int_{C-\epsilon}^C dC'' \frac{\partial c(C,C'')}{\partial C''} = \beta f'(C)(1-q)$$

The second integral needs to be split into two parts, with

$$\beta \int_C^1 dC'' f''(C'')X(C'')c(C,C'') \tag{30}$$
$$= x\beta \int_C^q dC'' f''(C'')c(C,C'') + \beta C \int_q^1 dC'' f''(C'')$$
$$= x\beta \int_C^q dC'' \frac{\partial}{\partial C''}[f'(C'')c(C,C'')] - x\beta \int_C^q dC'' f'(C'') \frac{\partial c(C,C'')}{\partial C''} + \beta C[f'(1) - f'(q)]$$
$$= x\beta[f'(q)c(C,q) - f'(C)c(C,C)] - x\beta \int_C^q dC'' f'(C'') \frac{\partial c(C,C'')}{\partial C''} + \beta C[f'(1) - f'(q)]$$
$$= x\beta[f'(q)C - f'(C)c(C,C)] - x\beta \int_C^q dC'' f'(C'') \frac{\partial c(C,C'')}{\partial C''} + \beta C[f'(1) - f'(q)]$$

Therefore we have

$$-(\mu_\infty - \beta f'(1) + (1-x)\beta f'(q))C$$
$$-(T + (1-q)(1-x)\beta f'(q))C$$

$$\frac{\partial}{\partial t} C(t,t') = -(\mu_\infty - \beta' f'(1) + (1-x)\beta f'(q))C(t,t') + \beta f'(C)(1-q)$$
$$- x\beta \int_{C(t,t')}^q dC'' f'(C'') \frac{\partial c(C(t,t'),C'')}{\partial C''}$$
$$= -(T + (1-q)(1-x)\beta f'(q))C(t,t') + \beta f'(C)(1-q) - x\beta \int_{C(t,t')}^q dC'' f'(C'') \frac{\partial c(C(t,t'),C'')}{\partial C''}$$

The asymptotic behavior is given by neglecting the integral, which will be the convolution of functions that vanish on each other's nonvanishing pieces, and noting then then $C(t,t') = 0$ is a solution. Therefore here we decay to zero. On the other limit $t' = t^*$, the integral again cancels. Assuming the derivative is much smaller than everything else, we have $0 = -(T + \beta(1-x)f'(q)(1-q))q + \beta(1-q)f'(q)$ or

$$\frac{q}{1-q} = \beta^2 f'(q)[1-(1-x)q] \tag{31}$$

which is the same condition as we found in the large time limit of the stationary regime. This means the solution is self-consistent, so far.

In order to fix the value of x, we need to examine the equation for the response function.

$$\begin{aligned}
\frac{\partial}{\partial t}\chi(t,t') &= -\mu_\infty \chi(t,t') + \int_{t'}^{t} dt''' \int_{t'''}^{t} dt''\, f''(C(t,t''))R(t'',t''')R(t,t'') \\
&= -\mu_\infty \chi(t,t') + \int_{t'}^{t} dt''\, f''(C(t,t''))\left[\int_{t'}^{t''} dt'''\, R(t'',t''')\right] R(t,t'') \\
&= -\mu_\infty \chi(t,t') - \int_{t'}^{t} dt''\, f''(C(t,t''))\chi(t'',t')R(t,t'') \\
&= -\mu_\infty \beta F(C(t,t')) + \beta^2 \int_{t'}^{t} dt''\, f''(C(t,t''))F(C(t'',t'))X(C(t,t''))\frac{\partial}{\partial t''}C(t,t'') \\
&= -\mu_\infty \beta F(C) + \beta^2 \int_{C}^{1} dC''\, f''(C'')F(c(C,C''))X(C'')
\end{aligned}$$

where in the second line we used causality to extend integration limits into regions where the response functions make the integrand zero. We want to evaluate this in the aging regime, where $C < q$. Again we divide the integral into sections, again noting that $c(C,C'') = C(t',t'')$ needs to have its stationary part treated differently. We have

$$\begin{aligned}
&\int_{C}^{1} dC''\, f''(C'')F(c(C,C''))X(C'') \\
&= x\int_{C}^{C+\epsilon} dC''\, f''(C'')F(c(C,C'')) + x\int_{C+\epsilon}^{q} dC''\, f''(C'')F(c(C,C'')) + F(C)\int_{q}^{1} dC''\, f''(C'')
\end{aligned} \tag{32}$$

The first integral takes an very small value, since it is the integral over a tiny region of a function that doesn't diverge in that region. Therefore we have

$$\begin{aligned}
&\int_{C}^{1} dC''\, f''(C'')F(c(C,C''))X(C'') \\
&= x\int_{C+\epsilon}^{q} dC''\, f''(C'')F(c(C,C'')) + F(C)[f'(1) - f'(q)]
\end{aligned} \tag{33}$$

At this point we have the following equation:

$$\frac{1}{\beta}\frac{\partial}{\partial t}\chi(t,t') = -\mu_\infty F(C) + \beta\left[x\int_{C+\epsilon}^{q} dC''\, f''(C'')F(c(C,C'')) + F(C)[f'(1)-f'(q)]\right]$$

In the aging regime, things move so slowly that time derivatives can be neglected. To move forward, we differentiate the whole equation with respect to C, remembering $F'(C) = -X(C)$. This gives

$$\begin{aligned}
0 &= -\mu_\infty F'(C) + \beta F'(C)[f'(1)-f'(q)] \\
&\quad + \beta x\left[-f''(C+\epsilon)F(c(C,C+\epsilon)) + \int_{C+\epsilon}^{q} dC''\, f''(C'')X(c(C,C''))\frac{\partial c(C,C'')}{\partial C}\right] \\
&= \mu_\infty x - \beta x[f'(1)-f'(q)] \\
&\quad + \beta x\left[-f''(C)F(q) + \int_{C+\epsilon}^{q} dC''\, f''(C'')X(c(C,C''))\frac{\partial c(C,C'')}{\partial C}\right] \\
&= \mu_\infty x - \beta x[f'(1)-f'(q)] - \beta x f''(C)(1-q) \\
&\quad + \beta x\int_{C+\epsilon}^{q} dC''\, f''(C'')X(c(C,C''))\frac{\partial c(C,C'')}{\partial C}
\end{aligned}$$

Now, we evaluate this at $C = q$. The integral vanishes because it is over an infinitesimal range, and we have (also dividing by x)

$$0 = \mu_\infty - \beta[f'(1)-f'(q)] - \beta f''(q)(1-q)$$

$$0 = T - \beta f'(q)[(1-x)q-1] - \beta f''(q)(1-q)$$

Recognizing our other condition, we can write

$$0 = T + T\frac{q}{1-q} - \beta f''(q)(1-q) = \frac{T}{1-q} - \beta f''(q)(1-q)$$

or

$$\beta^2 f''(q) = \frac{1}{(1-q)^2}$$

This is again the marginal edge of the stability condition for 1RSB! We see again that dynamics seems to care deeply about marginally stable 1RSB, for some reason. Combining the two equations, we have

$$\frac{qf''(q)}{f'(q)} = \frac{1-(1-x)q}{1-q}$$

which is solved to give

$$x = \frac{1-q}{q}\left(\frac{qf''(q)}{f'(q)} - 1\right) = \frac{1-q}{q}(p-2)$$

We can now evaluate quantities like μ_∞, which is

$$\begin{aligned}\mu_\infty &= T + \beta f'(1) - \beta(1-x)f'(q)q \\ &= T + \beta[f'(1) - f'(q) + (1-q)qf''(q)] \\ &= T + \frac{1}{2}p\beta\left[1 - q^{p-1} + (1-q)(p-1)q^{p-1}\right] \\ &= T + \frac{1}{2}p\beta\left[1 - q^p\left(1 - \frac{1-q}{q}(p-2)\right)\right]\end{aligned} \tag{34}$$

and likewise the asymptotic energy density is

$$E_\infty = -\frac{1}{2}\beta\left[1 - q^p\left(1 - \frac{(p-2)(1-q)}{q}\right)\right]$$

Here we see something interesting. We know that the conditions for stability first become satisfied for pure models when

$$q = \frac{p-2}{p-1} \qquad T = \sqrt{\frac{p(p-2)^{p-2}}{2(p-1)^{p-1}}}$$

For this q, E_∞ is the same as the energy density in the paramagnetic phase, $-\frac{1}{2}\beta$. However, between T_d and T_K, where the Boltzmann distribution sees no phase transition, the energy reached by asymptotic descent is systemically higher than the equilibrium energy density! Even the asymptotic dynamics does not ultimately equilibrate.

A few notes about the solution. If we look at the aging equation for the correlation, we see that neglecting the small derivative, it is solved by functions of the form $C(t, t') = \mathcal{C}(t'/t) = q(t'/t)^\gamma$. In fact, it is solved by many more functions: when the derivative is neglected, there is an approximate symmetry of the equations to reparameterizations of time, with $t \mapsto h(t)$ for monotonically increasing functions h. Given our treatment of the integrals, this should be obvious in hindsight: we were able to write the equations in such a way that time was not mentioned at all. Given this invariance, determining γ or other details of the solution requires careful (numeric) study of the aging region keeping the derivative. Such analysis reveals γ to not be universal, depending continuously on p and T.

Second, how can we interpret the generalized FDT involved in this solution; what is meaning of x? The basic picture is that two different timescales emerge in the system, one in which the system appears equilibrated in a region given by the overlap q, and one in which the system cannot reach equilibrium. In each, dynamics is governed by a different effective temperature. Since the slope of $F(C)$ decreases in the long-timescale regime, this means that the effective temperature of that regime is higher, with $T_{\text{eff}} = T/x$. What is the meaning of an effective temperature in this context? Temperature is usually defined as the inverse of the derivative of entropy with respect to energy, or

$$\frac{1}{T} = \frac{\partial S}{\partial E} \tag{35}$$

So, a higher effective temperature indicates that the entropy is changing more slowly with energy than would be expected by thermodynamic considerations.