8 Franz–Parisi potential

We have seen a variety of ways of analyzing a disordered system: the static Boltzmann distribution, the dynamic behavior in and out of equilibrium, and the complexity of metastable states. Here, we well introduce a final method of analysis. This is the Franz–Parisi potential, named for the researchers who introduced it. It is a method for analyzing systems that does not rely on the presence of disorder to average over, though in practice here we will still make such an average.

The potential is based on the following idea: what are the statistics of configurations restricted to lie a given overlap from a reference configuration drawn from the Gibbs ensemble? In particular, we will examine the free energy of such a restricted ensemble. A configuration is drawn from the Boltzmann distribution with probability distribution

$$p(\boldsymbol{\sigma}) = \frac{1}{Z} e^{-\beta H(\boldsymbol{\sigma})} \tag{1}$$

The free energy of an ensemble of configurations restricted to like an overlap q from this reference is

$$V(q \mid \boldsymbol{\sigma}) = -T \log\left(\int d\mathbf{s}\, e^{-\beta H(\mathbf{s})} \delta(Nq - \mathbf{s} \cdot \boldsymbol{\sigma})\right) \tag{2}$$

Averaging over the reference configuration, we have

$$\begin{aligned} V(q) &= \int d\boldsymbol{\sigma}\, p(\boldsymbol{\sigma}) V(q \mid \boldsymbol{\sigma}) \\ &= -T\frac{1}{Z}\int d\boldsymbol{\sigma}\, e^{-\beta H(\boldsymbol{\sigma})} \log\left(\int d\mathbf{s}\, e^{-\beta H(\mathbf{s})} \delta(Nq - \mathbf{s} \cdot \boldsymbol{\sigma})\right) \end{aligned} \tag{3}$$

In order to analyze the average of this potential over the disorder, we need to take care both of the logarithm and of the Z^{-1}. First, we introduce one set of $m-1$ replicas to treat the latter, which gives

$$\begin{aligned} V(q) &= -T \lim_{m\to 0} Z^{m-1} \int d\boldsymbol{\sigma}\, e^{-\beta H(\boldsymbol{\sigma})} \log\left(\int d\mathbf{s}\, e^{-\beta H(\mathbf{s})} \delta(Nq - \mathbf{s} \cdot \boldsymbol{\sigma})\right) \\ &= -T \lim_{m\to 0} \int \left(\prod_{a=1}^{m} d\boldsymbol{\sigma}_a\right) e^{-\beta \sum_{a=1}^{m} H(\boldsymbol{\sigma}_a)} \log\left(\int d\mathbf{s}\, e^{-\beta H(\mathbf{s})} \delta(Nq - \mathbf{s} \cdot \boldsymbol{\sigma}_1)\right) \end{aligned} \tag{4}$$

Note that crucially one of the $\boldsymbol{\sigma}$ replicas is special: $m-1$ of them came from the partition function Z and have to coupling to $\mathbf{s}$, while one of them comes from the numerator and therefore is present in the δ function. Next we need to treat the logarithm in the usual way, which gives

$$
\begin{aligned}
V(q) = -T \lim_{m\to 0} \lim_{n\to 0} \frac{\partial}{\partial n} \int \left(\prod_{a=1}^{m} d\boldsymbol{\sigma}_a\right) \left(\prod_{a=1}^{n} d\mathbf{s}_a\right) \\
\times\, e^{-\beta \sum_{a=1}^{m} H(\boldsymbol{\sigma}_a) - \beta \sum_{a=1}^{n} H(\mathbf{s}_a)} \prod_{a=1}^{n} \delta(Nq - \mathbf{s}_a \cdot \boldsymbol{\sigma}_1)
\end{aligned}
\tag{5}
$$

That's a whole lot of replicas! We are in a position to average over the Hamiltonian H. This gives

$$
\begin{aligned}
&\overline{e^{-\beta \sum_{a=1}^{m} H(\boldsymbol{\sigma}_a) - \beta \sum_{a=1}^{n} H(\mathbf{s}_b)}} \\
&= e^{\frac{1}{2}\beta^2 [\sum_{a=1}^{m}\sum_{b=1}^{m} \overline{H(\boldsymbol{\sigma}_a)H(\boldsymbol{\sigma}_b)} + 2\sum_{a=1}^{m}\sum_{b=1}^{n} \overline{H(\boldsymbol{\sigma}_a)H(\mathbf{s}_b)} + \sum_{a=1}^{n}\sum_{b=1}^{n} \overline{H(\mathbf{s}_a)H(\mathbf{s}_b)}]} \\
&= e^{\frac{N}{2}\beta^2 [\sum_{a=1}^{m}\sum_{b=1}^{m} f(\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b}{N}) + 2\sum_{a=1}^{m}\sum_{b=1}^{n} f(\boldsymbol{\sigma}_a\cdot\mathbf{s}_b) + \sum_{a=1}^{n}\sum_{b=1}^{n} f(\mathbf{s}_a\cdot\mathbf{s}_b)]}
\end{aligned}
\tag{6}
$$

This gives us three different overlap order parameters in our integral:

$$
Q_{ab} = \frac{1}{N}\mathbf{s}_a \cdot \mathbf{s}_b \qquad P_{ab} = \frac{1}{N}\boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b \qquad R_{ab} = \frac{1}{N}\mathbf{s}_a \cdot \boldsymbol{\sigma}_b \tag{7}
$$

that is, the overlap between configurations drawn from the restricted ensemble; the overlap between configurations drawn from equilibrium; and the overlap between the two. We can define these order parameters inside the integral with a δ-function. Following the standard set of steps, this changes the measure from

$$
\left(\prod_{b=1}^{n} d\mathbf{s}_b\right)\left(\prod_{a=1}^{m} d\boldsymbol{\sigma}_a\right) = dQ\, dP\, dR \left(\det \begin{bmatrix} Q & R \\ R^T & P \end{bmatrix}\right)^{\frac{N}{2}} \tag{8}
$$

We therefore have

$$
\begin{aligned}
V(q) = -T \lim_{m\to 0} \lim_{n\to 0} \frac{\partial}{\partial n} \int dQ\, dP\, dR\, e^{\frac{N}{2}\log\det(Q - RP^{-1}R^T) + \frac{N}{2}\log\det P} \\
\times\, e^{\frac{N}{2}\beta^2[\sum_{a=1}^{m}\sum_{b=1}^{m} f(P_{ab}) + 2\sum_{a=1}^{m}\sum_{b=1}^{n} f(R_{ba}) + \sum_{a=1}^{n}\sum_{b=1}^{n} f(Q_{ab})]} \\
\times \prod_{a=1}^{n} \delta(q - R_{a1})
\end{aligned}
\tag{9}
$$

where we have used an identity for the determinant of a block matrix. Now we have an effective action in terms of *three* order parameter matrices: the overlaps between the reference configurations P which is $m \times m$, the overlaps between the constrained configurations Q which is $n \times n$, and the overlaps between reference configurations and constrained configurations R which is $n \times m$. R is not completely free however: its first column must be made up of overlaps q.

First, we can make a large simplification: because of the structure of this kind of problem, the saddle-point conditions for the overlaps P between the reference configurations σ do not depend on the rest of the order parameters in the $m, n \to 0$ limit. Qualitatively, this should make sense: we constructed the potential so that σ was drawn independently of the other configurations, and therefore its properties should likewise be independent. In fact, we know what they should be: P should precisely equal the matrix of overlaps Q found for an equilibrium configuration of the model. We can see this by writing

$$V(q) = -T \lim_{m\to 0}\lim_{n\to 0}\frac{\partial}{\partial n}\int dQ\,dP\,dR\,e^{Nm\mathcal{S}_m(P)+Nn\mathcal{S}_{n,m}(Q,R|P)}\prod_{a=1}^{n}\delta(q-R_{a1})$$

for effective actions

$$\mathcal{S}_m(P) = \frac{1}{2m}\left[\beta^2\sum_{ab}^{m} f(P_{ab}) + \log\det P\right] \tag{10}$$

which is just the effective action for the equilibrium free energy, and

$$\mathcal{S}_n(Q, R \mid P) = \frac{1}{2n}\left[\beta^2\sum_{ab}^{n} f(Q_{ab}) + \beta^2\sum_{a=1}^{n}\sum_{b=1}^{m} f(R_{ab}) + \log\det(Q - RP^{-1}R^T)\right]$$

where $R_{a1} = q$ is enforced for all a. Then, the saddle-point equation for P is

$$0 = \frac{\partial\mathcal{S}_m(P)}{\partial P} + \frac{n}{m}\frac{\partial\mathcal{S}_n(Q, R \mid P)}{\partial P} \tag{11}$$

Therefore, P just takes the saddle-point value P^* for the equilibrium system at the same temperature. Why doesn't this argument work in reverse: shouldn't it be equally true that Q doesn't depend on P and R, if we write the actions in the reverse way? The answer is *no*, and this is because of the structure of R. In a RS or RSB ansatz, R will have constant columns, and therefore every of its n rows will be the same. Therefore, in the limit of n to zero at finite m, the coupling that R induces between P and Q vanishes. However, the opposite is not true: at least the first column of R is different from the other $m-1$, because it is fixed by δ functions. Therefore, as m goes to zero at finite n, the first column contributes with a prefactor 1 and the rest of the matrix with a prefactor $m - 1 = -1$.

Assuming we take P to have its saddle-point value, the action $\mathcal{S}_m$ will not effect the value of $V(q)$ in the twin limits, since there is no derivative with respect to m in the expression. Therefore, we can write

$$V(q) = -T \lim_{m\to 0}\lim_{n\to 0}\frac{\partial}{\partial n}\int dQ\,dR\,e^{Nn\mathcal{S}_{n,m}(Q,R|P^*)} \tag{12}$$

We now have to make suppositions about the form of the matrices Q and R. In general, if P is 1RSB then we expect the same structure of Q. However, for simplicity and because it is the regime we are interested in studying, we will assume P and therefore Q are both replica symmetric. In the replica symmetric phase, all configurations $\boldsymbol{\sigma}_b$ for $b > 1$ are completely uncorrelated from $\boldsymbol{\sigma}_1$. Therefore, we can divide the overlaps R between non-reference equilibrium states $\boldsymbol{\sigma}_b$ and constrained states $\mathbf{s}_a$ into just two values, with the form of the matrix

$$R = \begin{bmatrix} q & r & \cdots & r \\ \vdots & \vdots & \ddots & \vdots \\ q & r & \cdots & r \end{bmatrix} \tag{13}$$

with n rows and m columns. R only enters the effective action via $RP^{-1}R^T$, and when P is RS $P = I$ and

$$\begin{bmatrix} q & r & \cdots & r \\ \vdots & \vdots & \ddots & \vdots \\ q & r & \cdots & r \end{bmatrix} \begin{bmatrix} q & \cdots & q \\ r & \cdots & r \\ \vdots & \ddots & \vdots \\ r & \cdots & r \end{bmatrix} = (q^2 + (m-1)r^2) \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \tag{14}$$

a factor multiplying the constant matrix. Note that the constant matrix is itself a RS matrix, with the same value on and off the diagonal. If we call the off-diagonal of Q q_0 to differentiate it from the constrained overlap q, $Q - RR^T$ is a replica symmetric matrix with diagonal $1 - q^2 + (1-m)r^2$ and off-diagonal $q_0 - q^2 + (1-m)r^2$. This gives

$$\lim_{m\to 0}\lim_{n\to 0}\frac{1}{n}\log\det(Q - RR^T) = \frac{q_0 - q^2 + r^2}{1-q_0} + \log(1-q_0) \tag{15}$$

We therefore have, in the twin limits,

$$\mathcal{S}(q_0, r \mid q) = \frac{1}{2}\left[\beta^2(f(1) - f(q_0)) + 2\beta^2(f(q) - f(r)) + \frac{q_0 - q^2 + r^2}{1-q_0} + \log(1-q_0)\right] \tag{16}$$

The equation for r is $\beta^2 f'(r) = \frac{r}{1-q_0}$, which is solved only by $r = 0$. The equation for q_0 is

$$\beta^2 f'(q_0) = \frac{q_0 - q^2}{(1-q_0)^2} \tag{17}$$

Solving this equation for its extremal value q_0^* and substituting it back into the action gives the potential, with

$$V(q) = -TN\mathcal{S}(q_0^*, 0 \mid q) \tag{18}$$

What does this look like in practice? The potential always has a minimum at $q = 0$, then raises from there. Depending on the temperature, it either continues to grow monotonically (high temperatures) or has another minimum at finite q (low temperatures). We can study analytically the properties of this other minima, when it exists. Such a minima would occur for

$$\frac{\partial V(q)}{\partial q} = -T\frac{\partial \mathcal{S}}{\partial q} - T\frac{\partial \mathcal{S}}{\partial q_0}\frac{\partial q_0}{\partial q} = -T\frac{\partial \mathcal{S}}{\partial q} \tag{19}$$

where we have used the fact that $\partial\mathcal{S}/\partial q_0$ is zero because of the saddle-point condition. This gives the condition

$$0 = \beta^2 f'(q) - \frac{q}{1-q_0} \tag{20}$$

If we try $q_0 = q$, we see that this equation and the saddle point equation for q_0 both match, thereby solving them. Not only does it solve the conditions, it does so with exactly the equilibrium saddle-point condition on q for $m = 1$. Therefore, any extremum of $V(q)$ corresponds to a saddle-point value of $q_0 = q$, and the value of q at this point is also the (potentially unstable) equilibrium value. In fact, we can understand the properties of extrema of $V(q)$ by studying instead the simpler explicit function

$$V^*(q) = -T\mathcal{S}(q, 0 \mid q) = -\frac{T}{2}\left[\beta^2(f(1) + f(q)) + q + \log(1-q)\right] \tag{21}$$

which is in general not equal to $V(q)$ but which, from the previous argument, has exactly the same minima and maxima as $V(q)$. A natural question is: at which temperature does tho second minima appear? We can answer this by asking instead at which temperature it *disappears*, which is equivalent to the point where the maximum and minimum merge, where the curvature vanishes. This condition is

$$0 = \frac{\partial^2 V^*(q)}{\partial^2 q} \propto \beta^2 f''(q) - \frac{1}{(1-q)^2} \tag{22}$$

which is exactly the marginal condition. Combined with the condition above that q takes its equilibrium value, we see that the second minimum emerge exactly at the temperature of the dynamic transition. The metastable states responsible for that dynamic transition correspond in this potential picture with a metastable minimum.

The place where the second minimum becomes the absolute minimum of V is also notable. To identify this point, we need to compare the value of V at zero and at the second minimum. This condition gives

$$0 = V^*(q^*) - V^*(0) = -\frac{T}{2}\left[\beta^2 f(q^*) + q^* + \log(1-q^*)\right] \tag{23}$$

But this is exactly the saddle point condition for m at $m = 1$, which we used in a previous lecture to identify the spin glass transition temperature! Therefore, the point at which the nonzero q minimum switches from metastable to stable is precisely the static transition temperature.

Finally, recall that the complexity of states with free energy F is given by

$$\Sigma = x\beta F + \frac{1}{2}\left[\beta^2 x[f(1) - (1-x)f(q)] + x\log(1-q) + \log\frac{1-(1-x)q}{1-q}\right] \tag{24}$$

for q determined by

$$\beta^2 f'(q) = \frac{q}{(1-q)(1-q(1-x))} \tag{25}$$

and x given by

$$x = \frac{1}{f(q)}\left[-TF - \frac{1}{2}\left(f(1) - f(q) + (1-q)f'(q) + T^2\log(1-q)\right)\right] \tag{26}$$

Note that when $x = 1$, we are counting states that satisfy the equilibrium equation for $m = 1$. We can call these 'equilibrium' states even though above the static transition temperature they are not really equilibrium states. The condition $x = 1$ corresponds to a free energy

$$F = -\frac{1}{2}\beta\left(f(1) + f(q) + (1-q)f'(q) + T^2\log(1-q)\right) \tag{27}$$

and therefore a complexity

$$\begin{aligned}\Sigma &= \beta F + \frac{1}{2}\beta^2 f(1) \\ &= -\frac{1}{2}\beta^2\left(f(q) + (1-q)f'(q) + T^2\log(1-q)\right) \\ &= -\frac{1}{2}\left[\beta^2 f(q) + q + \log(1-q)\right]\end{aligned} \tag{28}$$

where in the last line we used the equilibrium saddle-point condition for q to replace the term with $f'(q)$ in it. Comparing with what we have above, this gives $\Sigma_{eq} = \beta[V(q^*) - V(0)]$ – that is, the complexity of metastable states that meet the equilibrium condition on q is proportional to the difference in this potential between the two minima. Here, we see very explicitly that the dynamic transition takes place when a complexity of metastable states first appears, and that the static transition takes place when the complexity of equilibrium states reaches zero.